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CONGRUENCES BETWEEN CUSP FORMS: THE (p, p) CASE

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1. Introduction. It has been known for some time, as a consequence of the work of numerous mathematicians, that newforms for congruence subgroups of $SL_2(\mathbb{Z})$ give rise to a compatible system of ℓ-adic representations, and if the p-adic representations attached to two newforms are isomorphic for any prime p, then the newforms are, in fact, equal. But the corresponding statement is not true for the mod p reductions of p-adic representations attached to newforms, as different newforms can give rise to isomorphic mod p representations which arise from reduction mod p of the corresponding p-adic representations. (This is well defined if we assume that the mod p representation is absolutely irreducible.) This is a reflection of the fact that distinct newforms can be congruent modulo p. To study the different levels from which a given modular mod p representation can arise is interesting and has been much studied.

Thus, if we consider the image of the classical Hecke operators in the ring of endomorphisms of the Jacobian $J_0(S)$ of the modular curve $X_0(S)$ for some integer S, then the resulting $\mathbb{Z}$-algebra is of finite rank over $\mathbb{Z}$. We denote it by $\mathbb{T}_S$. Then to any maximal ideal m of $\mathbb{T}_S$ of residue characteristic, say p, we may attach, after the work of Eichler-Shimura, a semisimple representation:

$$\rho_m\colon \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{GL}_2(\overline{\mathbb{T}_{\mathbb{S}}/m}),$$

such that it is unramified at all primes r prime to pS, and for such primes $\mathrm{tr}(\rho_m(\mathrm{Frob}_r))$ is the image of T_r in $\mathbb{T}_{\mathbb{S}}/m$ and $\det(\rho_m(\mathrm{Frob}_r)) = r$. We study only such representations which are also absolutely irreducible. On viewing ρ_m abstractly, one may try to classify all the pairs $(\mathbb{T}_M, n)$, where n is a maximal ideal of $\mathbb{T}_M$, that give rise (in the above fashion) to a representation isomorphic to ρ_m in a nontrivial way (i.e., n should be associated to a newform of level M). This classification has been essentially carried out in the work of several people—Mazur, Ribet, Carayol, Diamond, and Taylor—for all M prime to p. In this paper, we study the case when we do not impose this condition. We shall talk colloquially of this as the (p, p) case and will assume that $p \geqslant 5$.

This case differs in many salient points. It follows from the classification of Carayol in [C] that the exponent with which any prime ℓ different from p occurs in the factorisation of any M as above is bounded. As a consequence of a more precise result, which we prove in this paper as Theorem 2, we see that arbitrarily

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