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MAXIMALITY OF GALOIS ACTIONS FOR COMPATIBLE SYSTEMS

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§0. Introduction. The maximality of Galois groups associated with cohomology of varieties was first discussed by Serre [18], [19]. He proved that if K is a number field and E/K an elliptic curve not potentially of CM-type, then for all $\ell \gg 0$, the homomorphism

$$\mathrm{Gal}(\bar{K}/K) \to \mathrm{GL}(T_\ell(E)),$$

giving the Galois action on the Tate module of E, is surjective. We would like to generalize this to the case of a compatible system of n-dimensional representations

$$\rho_\ell \colon \mathrm{Gal}(\bar{K}/K) \to \mathrm{GL}_n(\mathbb{Q}_\ell),$$

in the sense of Serre [18]. Of course, the image of ρ_ℓ is not generally Zariski-dense in $\mathrm{GL}_n(\mathbb{Q}_\ell)$, so the maximality condition must be formulated relative to the Zariski closure, G_ℓ, of $\rho_\ell(\mathrm{Gal}(\bar{K}/K))$. One might hope that the image of ρ_ℓ is a maximal compact subgroup of $G_\ell(\mathbb{Q}_\ell)$, but this is too optimistic: the center causes problems, and we do not even know a priori that G_ℓ is reductive. To formulate a maximality conjecture that avoids such problems, we introduce maps

$$G_\ell^\circ \xrightarrow{\sigma} G_\ell^{\mathrm{ad}} \xleftarrow{\tau} G_\ell^{\mathrm{sc}},$$

where G_ℓ° denotes the identity component of G_ℓ, G_ℓ^{ad} the quotient of G_ℓ° by its radical, and G_ℓ^{sc} the simply connected cover of G_ℓ^{ad}. We expect that for $\ell \gg 0$,

$$\tau^{-1}(\sigma(\rho_\ell(\mathrm{Gal}(\bar{K}/K)) \cap G_\ell^\circ(\mathbb{Q}_\ell))) \tag{0.1}$$

should be a *hyperspecial* maximal compact subgroup of G_ℓ^{sc}. This technical condition implies, in particular, that with respect to Haar measure, the group (0.1) is of maximal volume. The main result of this paper is a weaker claim, namely that (0.1) is a hyperspecial maximal compact for a set of primes ℓ of Dirichlet density 1. The argument is valid not only for systems of cohomology representations, but for all compatible systems of ℓ-adic Galois representations.

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