Vol. 80, No. 3 DUKE MATHEMATICAL JOURNAL December 1995

SHARP INEQUALITIES FOR MARTINGALES WITH APPLICATIONS TO THE BEURLING-AHLFORS AND RIESZ TRANSFORMS

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§0. Introduction. The purpose of this paper is to prove some sharp inequalities for martingales, and from these obtain new information on the L^p-constants for Riesz transforms, composition of two Riesz transforms and for the Beurling-Ahlfors operator in the complex plane $\mathbb{C}$. The latter operator is the 2-dimensional analogue of the classical Hilbert transform, and it plays a fundamental role (e.g., see [1], [2], [11], [12]) in the study of quasiconformal mappings, partial differential equations, complex analysis, and, as shown recently by Iwaniec and Martin ([13], [14]) in the study of certain singular integrals in $\mathbb{C}^n$ and on differential forms with even kernels. We will first describe the martingale results.

Let $(\Omega, \mathscr{F}, P)$ be a probability space and $\mathscr{F} = \{\mathscr{F}_t\}_{t \geqslant 0}$ be a nondecreasing family of sub-σ-fields of $\mathscr{F}_\infty$. For any two real-valued martingales X and Y with respect to $\mathscr{F}$, we say that X is orthogonal to Y if the quadratic covariation between X and Y, denoted by $\langle X, Y \rangle_t$, is 0 for all $t \geqslant 0$. Motivated by Burkholder ([6], [7]) we shall also say that Y is differentially subordinate to X if the quadratic variation of X minus that of Y, $\langle X \rangle_t - \langle Y \rangle_t$, is a nondecreasing function of t for $t \geqslant 0$. Unless otherwise indicated, we also assume throughout the paper that $X_0 = Y_0 = 0$. The same definition for differential subordination applies if both X and Y are $\mathbb{H}$-valued martingales where $\mathbb{H}$ is a separable Hilbert space over $\mathbb{R}$, or if one martingale is $\mathbb{H}$-valued and the other is real-valued. The constants that we obtain do not depend on the Hilbert space, so we could just as well assume $\mathbb{H} = \mathbb{R}^d$, for any positive integer d. We say that two $\mathbb{R}^d$-valued martingales are orthogonal if $\langle X_i, Y_j \rangle = 0$ for all $1 \leqslant i, j \leqslant d$, where $(X_1, \ldots, X_d)$, $(Y_1, \ldots, Y_d)$ are coordinates of X and Y.

For any $1 < p < \infty$, we define

$$p^* = \max\left\{p, \frac{p}{p-1}\right\},$$

$$C_p = \begin{cases} \tan\left(\dfrac{\pi}{2p}\right), & 1 < p \leqslant 2 \\ \cot\left(\dfrac{\pi}{2p}\right), & 2 \leqslant p < \infty \end{cases},$$

Received 11 July 1994. Revision received 20 April 1995.
Bañuelos supported in part by the National Science Foundation.
Wang supported in part by a Summer Research Grant of DePaul University.