Vol. 78, No. 2 DUKE MATHEMATICAL JOURNAL © May 1995

PROJECTIVE STRUCTURES INDUCING COVERING MAPS

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1. Introduction. Similar to the way a complex analytic structure can be defined on a two-dimensional manifold S, we define a complex projective structure on S as a certain equivalent class of projective coordinate coverings $\{(U_\alpha, z_\alpha)\}_\alpha$ over S. Here, $\{(U_\alpha, z_\alpha)\}_\alpha$ is a projective coordinate covering by definition if the transition functions $f_{\alpha\beta}: z_\alpha \to z_\beta$ on $U_\alpha \cap U_\beta$ are complex projective mappings = Möbius transformations. A projective structure determines a unique complex structure on S; we say that the projective structure is subordinate to this complex structure. In general, a Riemann surface S admits distinct projective structures subordinate to the complex structure.

We consider parametrization of the projective structures on a compact Riemann surface S of genus $\geqslant 2$. Let $\pi: \tilde{U} \to S$ be the universal cover of S with the covering transformation group $\tilde{\Gamma}$. The pull back of a projective structure on S determines a projective structure on $\tilde{U}$, and the analytic continuation of the projective coordinate coverings defines a local homeomorphism $\tilde{f}: \tilde{U} \to \hat{\mathbf{C}}$ globally, which is called the developing map. Providing the complex structure for $\tilde{U}$, we may regard it as the unit disk $U = \{z \mid |z| < 1\}$. Then the developing map becomes a meromorphic local homeomorphism $f: U \to \hat{\mathbf{C}}$ and $\tilde{\Gamma}$ becomes a Fuchsian group Γ acting on U. For each element $\gamma \in \Gamma$, f differs from $f \circ \gamma$ by the left composition of a complex projective mapping m. Hence a homomorphism $\chi: \Gamma \to \text{Möb} = \{\text{Möbius transformations}\}$ is defined by the assignment $\chi(\gamma) = m$. This homomorphism is called the monodromy representation of the projective structure. It is clear that f_1 and f_2 are developing maps of the same projective structure if and only if there is a Möbius transformation h such that $h \circ f_1 = f_2$. In this case, the corresponding monodromy representations χ_1 and χ_2 satisfy $h \circ \chi_1 \circ h^{-1} = \chi_2$, and they are called equivalent. Thus a projective structure determines an equivalent class of the monodromy representations. Conversely, a classical result due to Poincaré says that this class determines uniquely the projective structure whenever S is compact. Hence, the projective structures are parametrized in $\mathrm{Hom}(\tilde{\Gamma}, \text{Möb})$, the space of all homomorphisms of $\tilde{\Gamma}$ into Möb, via their monodromy representations. Note that they hold a large part in $\mathrm{Hom}(\tilde{\Gamma}, \text{Möb})$. Indeed, any non-elementary homomorphism $\chi \in \mathrm{Hom}(\tilde{\Gamma}, \text{Möb})$ that splits into $\tilde{\Gamma} \to \mathrm{SL}(2, \mathbf{C}) \to \mathrm{PSL}(2, \mathbf{C}) \cong \text{Möb}$ comes from a projective structure (Kapovich [6]).

Another way to parametrize the projective structures relies on holomorphic quadratic differentials on S. Let $f: U \to \hat{\mathbf{C}}$ be the developing map of a projective

Received 24 May 1994. Revision received 22 November 1994.