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ON THE HYPERBOLICITY OF THE COMPLEMENTS OF CURVES IN ALGEBRAIC SURFACES: THE THREE-COMPONENT CASE

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1. Introduction. In complex analysis, hyperbolic manifolds have been studied extensively, with close relationships to other areas (cf., e.g., [20]). Hyperbolic manifolds are generalizations of hyperbolic Riemann surfaces to higher dimensions. Despite the fact that the general theory of hyperbolic manifolds is well developed, only very few classes of hyperbolic manifolds are known. But one could hope that "most" of the pseudoconvex quasi-projective varieties are in fact hyperbolic, provided that the degrees involved are high enough. In particular, it is believed that, e.g., the complements of most hypersurfaces in $\mathbb{P}_n$ are hyperbolic, if only their degree is at least $2n + 1$. More precisely, according to Kobayashi [18] and later Zaidenberg [30], one has the following.

CONJECTURE. *Let $\mathscr{C}(d_1, \ldots, d_k)$ be the space of k-tuples of hypersurfaces $C = (C_1, \ldots, C_k)$ in $\mathbb{P}_n$, where $\deg(C_i) = d_i$. Then, for all $(d_1, \ldots, d_k)$ with $\sum_{i=1}^k d_i =: d \geqslant 2n + 1$, the set $\mathscr{H}(d_1, \ldots, d_k) = \{C \in \mathscr{C}(d_1, \ldots, d_k)\colon \mathbb{P}_n \backslash \bigcup_{i=1}^k C_i$ is complete hyperbolic and hyperbolically embedded$\}$ contains the complement of a proper algebraic subset of $\mathscr{C}(d_1, \ldots, d_k)$.*

In this paper we shall restrict ourselves to the two-dimensional case. However, we consider also more general quasi-projective complex surfaces than the complements of curves in the projective plane.

Concerning the above conjecture, the following is known: it seems that the conjecture is the more difficult the smaller k is. Other than in the case of five lines ($\mathscr{C}(1, 1, 1, 1, 1)$), the conjecture was proved by M. Green in [15] in the case of a curve C consisting of one quadric and three lines ($\mathscr{C}(2, 1, 1, 1)$). Furthermore, it was shown for $\mathscr{C}(d_1, \ldots, d_k)$, whenever $k \geqslant 5$, by Babets in [3]. A result which went much further was given by Eremenko and Sodin in [9], where they proved a second main theorem of value distribution theory in the situation $k \geqslant 5$. Green

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