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GROUP COHOMOLOGY CONSTRUCTION OF THE COHOMOLOGY OF MODULI SPACES OF FLAT CONNECTIONS ON 2-MANIFOLDS

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1. Introduction. Let K be a compact connected semisimple Lie group, and let β be an element in the center $Z(K)$. If Σ is a closed 2-manifold of genus $g \geqslant 2$ with fundamental group Π, and $\Sigma_0 = \Sigma - D^2$, then $\pi_1(\Sigma_0) = \mathbb{F}$ where $\mathbb{F}$ is the free group on $2g$ generators. There is an associated moduli space of representations $\mathscr{M}_\beta = Y_\beta/K$ where $Y_\beta = \{\rho \in \mathrm{Hom}(\mathbb{F}, K): \rho(R) = \beta\}$ and K acts on $\mathrm{Hom}(\mathbb{F}, K)$ by conjugation. Here R is the element of $\mathbb{F}$ corresponding to a loop winding once around the boundary $\partial\Sigma_0$. Because β is in the center $Z(K)$, each point in the space $\mathscr{M}_\beta$ gives rise to a representation of the fundamental group Π of the closed surface Σ into the group $K_c = K/Z(K)$.

The space $\mathscr{M}_\beta$ has two alternative descriptions. Via the holonomy map, $\mathscr{M}_\beta$ may be identified with the space of gauge equivalence classes of flat connections on a principal K bundle over Σ_0, for which the holonomy around the boundary $\partial\Sigma_0$ is the element β. We obtain a second alternative description once we fix a complex structure on Σ: then $\mathscr{M}_\beta$ becomes identified with a space of semistable holomorphic vector bundles (of prescribed rank and degree) over Σ.

Atiyah and Bott worked in this holomorphic setting and described the generators of the cohomology ring of $\mathscr{M}_\beta$ in terms of a holomorphic vector bundle $\mathbb{U}$ over $\mathscr{M}_\beta \times \Sigma$ (the *universal bundle*), whose restriction to a point $m \in \mathscr{M}_\beta$ is the holomorphic vector bundle over Σ corresponding to m. The Künneth decomposition of the Chern classes of $\mathbb{U}$ yields classes in $H^*(\mathscr{M}_\beta)$, which are the generators of the cohomology ring. The purpose of the present paper is to give an explicit description of these generators in a representation-theoretic setting, using group cohomology.

Our starting point is the paper [W] of Weinstein: the construction we present below generalizes Weinstein's construction of the symplectic form on moduli space, whose cohomology class is one of the generators described above. Goldman [G] constructed the symplectic form using group cohomology, but his proof that this form was closed used its gauge-theory description. Karshon [K] gave the first group cohomology proof that the symplectic form was closed. In [W], Weinstein interpreted Karshon's construction in terms of the realization of $H^*(BK)$ via the de Rham cohomology of simplicial manifolds, which is due to Bott and

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