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RAGHUNATHAN'S CONJECTURES FOR CARTESIAN PRODUCTS OF REAL AND p-ADIC LIE GROUPS

MARINA RATNER

To Professor Armand Borel

Introduction. The problem of extending Raghunathan's Conjectures to cartesian products of *algebraic* groups over local fields of characteristic zero (this is referred to as the $\mathbb{S}$-arithmetic setting) was raised by A. Borel and G. Prasad in [BP] (see also [P]). They showed that if $\mathbb{S}$ is a finite set of places (containing the archimedean ones) of a number field κ and $\kappa_{\mathbb{S}}$ the direct sum of the completions κ_{s} of κ at $s \in \mathbb{S}$, then the set of values of a nondegenerate isotropic irrational quadratic form over $\kappa_{\mathbb{S}}$ at $\mathbb{S}$-integral points is not discrete around the origin. They also pointed out that the validity of Raghunathan's conjecture on orbit closures for the $\mathbb{S}$-arithmetic case (see Theorem 2 below) would imply that this set is dense in $\kappa_{\mathbb{S}}$. These two results for *real* quadratic forms are equivalent and constitute the content of the Oppenheim conjecture proved by Margulis in [M2].

It turns out that the ideas and methods developed in [R1]–[R4] for *real* Lie groups can be applied to prove Raghunathan's Conjectures for a more *general* case than the $\mathbb{S}$-arithmetic setting, namely, cartesian products of real and p-adic Lie groups. (A Lie group over a local field of characteristic zero can be viewed as either a real Lie group or a p-adic Lie group.) This generality is largely responsible for the size of this paper. Also the literature on general p-adic Lie groups is rather scarce and we needed to develop some necessary theory of such groups. (See Sections 1, 2, and 4 and more.)

More specifically, let $\mathbf{G}$ be a locally compact group (all topological groups in this paper are assumed to be second countable), $\mathbf{\Gamma}$ a discrete subgroup of $\mathbf{G}$, and $\pi: \mathbf{G} \to \mathbf{\Gamma}\backslash\mathbf{G}$ the covering projection $\pi(\mathbf{g}) = \mathbf{\Gamma g}$, $\mathbf{g} \in \mathbf{G}$. The group $\mathbf{G}$ acts by right translations on $\mathbf{\Gamma}\backslash\mathbf{G}$: $x \to x\mathbf{g}$, $x \in \mathbf{\Gamma}\backslash\mathbf{G}$, $\mathbf{g} \in \mathbf{G}$. The group $\mathbf{\Gamma}$ is called a *lattice* in $\mathbf{G}$ if there is a *finite* $\mathbf{G}$-invariant Borel measure on $\mathbf{\Gamma}\backslash\mathbf{G}$.

A subset $A \subset \mathbf{\Gamma}\backslash\mathbf{G}$ is called *homogeneous* if there is an $x \in \mathbf{\Gamma}\backslash\mathbf{G}$ and a closed subgroup $\mathbf{H} \subset \mathbf{G}$ such that $\mathbf{xHx}^{-1} \cap \mathbf{\Gamma}$ is a lattice in $\mathbf{xHx}^{-1}$, $\mathbf{x} \in \pi^{-1}\{x\}$, and $A = x\mathbf{H}$. In this case $A = x\mathbf{H}$ is a closed subset of $\mathbf{\Gamma}\backslash\mathbf{G}$ and there is an $\mathbf{H}$-invariant Borel probability measure $\nu_{\mathbf{H}}$ or $\mathbf{\Gamma}\backslash\mathbf{G}$ supported on $x\mathbf{H}$.

Let μ be a Borel probability measure on $\mathbf{\Gamma}\backslash\mathbf{G}$. Define

$$\Lambda(\mu) = \{\mathbf{g} \in \mathbf{G}\text{: the action of } \mathbf{g} \text{ on } \mathbf{\Gamma}\backslash\mathbf{G} \text{ preserves } \mu\}.$$

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