Vol. 77, No. 1 DUKE MATHEMATICAL JOURNAL © January 1995

GEOMETRIC CONDITIONS AND EXISTENCE OF BI-LIPSCHITZ PARAMETERIZATIONS

TATIANA TORO

1. Introduction. Reifenberg [R] and Semmes [S1], [S2] have studied how geometric conditions on a set Σ, specifically good approximations by affine spaces, imply the existence of good parameterizations. Reifenberg's condition, roughly speaking, says that if in a neighborhood of a point ζ_0, Σ can be approximated by affine spaces at every point and at every scale, then there exists a neighborhood of ζ_0 that is homeomorphic to a disc. In this paper we refine Reifenberg's condition in order to guarantee the existence of bi-Lipschitz parameterizations at least locally. In particular this gives a partial answer to a question formulated by Semmes concerning the existence of bi-Lipschitz parameterizations for chord-arc surfaces with small constant (CASSC). In order to be more specific we need to introduce some definitions.

Definition 1.1. A locally compact set Σ of Hausdorff dimension n in $\mathbf{R}^{n+m}$ is said to satisfy Reifenberg's (δ, R) condition at $\zeta_0 \in \Sigma$ if, for every $\zeta \in \Sigma \cap B(R, \zeta_0)$ and every $r \in (0, R]$, there corresponds an n-dimensional plane $L(r, \zeta)$ containing ζ such that

$$\frac{1}{r} D[\Sigma \cap B(r, \zeta), L(r, \zeta) \cap B(r, \zeta)] \leqslant \delta,$$

where D denotes the Hausdorff distance.

THEOREM (Reifenberg [R], [M, 10.5]). *There exists $\delta > 0$ depending only on n and m so that if Σ satisfies the (δ, R) condition at ζ_0, then $\Sigma \cap B(R/32, \zeta_0)$ is a topological disc.*

Assuming that $\Sigma \subset \mathbf{R}^{n+m}$ is a locally compact set of Hausdorff dimension n satisfying Reifenberg's (δ, R) condition at ζ_0, we write

$$\theta(r, \zeta) = \inf_L \left\{ \frac{1}{r} D[\Sigma \cap B(r, \zeta), L \cap B(r, \zeta)] \right\},$$

where the infimum is taken over all n-planes containing ζ. Let $L(r, \zeta)$ be an n-plane satisfying

Received 27 May 1993.
The author was supported by NSF grant DMS-9100383.