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EQUIVALENCE OF THE DARBOUX AND GARDNER METHODS FOR INTEGRATING HYPERBOLIC EQUATIONS IN THE PLANE

DAVID HARTLEY, ROBIN W. TUCKER, AND PHILIP A. TUCKEY

Introduction. In a well-posed Cauchy problem for a partial differential equation (PDE) system, the solution is uniquely determined by the Cauchy data. Nevertheless, the construction of that solution generally involves further PDE systems. If these latter systems cannot be solved, then the construction of the solution for the original Cauchy problem falters, even though that solution is known to exist and to be unique. It is therefore interesting to ask under what conditions a Cauchy problem may be solved using ordinary differential equation (ODE) systems, since these are always formally integrable locally.

Several approaches to the integration of partial differential equations by such methods are available. This paper focuses on two: the Darboux method [1] and the stability analysis of Gardner [2], [3], [4].

Darboux's method is a generalisation of techniques developed by Monge and Ampère that construct additional equations, called intermediate integrals, which are compatible with the original PDE. Taken together, the original PDE and the intermediate integral may be solved by ODE systems.

Gardner's method takes a different approach that constructs a vector field, in a suitable space, which satisfies a certain stability property in relation to the PDE system. The integral curves of this vector field then develop the Cauchy data into a full solution.

The main result of this paper is that for a hyperbolic second order PDE in one dependent and two independent variables, the Darboux and Gardner methods are equivalent. This result is presented in the context of solving the Cauchy problem. It is interesting to note that the equivalence of the two methods does not apply to exactly the same PDE system for the hyperbolic equation. Rather, the Gardner method for an arbitrary prolongation of a PDE system is equivalent to the Darboux method for its next prolongation.

In this paper, the language of exterior differential systems [5] is used to present and derive the results. First, in Section 1, the exterior systems formulation of the hyperbolic PDE is described, and some of the concepts and notation to be used in the remaining sections are introduced. Section 2 presents the Gardner method, essentially as described in [2]. A formulation of the Darboux method in the language of exterior systems is given in Section 3, and is applied specifically to

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