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ASYMPTOTIC SHAPE OF CUSP SINGULARITIES IN CURVE SHORTENING

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1. Introduction. In this paper we consider a smooth solution $\{\gamma(t): 0 < t < T\}$ of *curve shortening* which becomes singular at time $t = T$. We assume that $\gamma(t)$ is convex, i.e., has no inflection points, and that $\gamma(t)$ is symmetric with respect to reflection in the x-axis.

In [A] it was shown that the curvature k of $\gamma(t)$ remains bounded outside a finite number of singular arcs; on each of these singular arcs the normal angle θ turns through at least 180°. We assume that our solution $\gamma(t)$ has only one such singular arc, and that the normal θ turns through exactly 180°.

1.1. THEOREM. *Under the above hypotheses, the maximal curvature on $\gamma(t)$ blows up at the following rate:*

$$\max_{\gamma_t} k = (1 + o(1))\sqrt{\frac{\ln \ln(1/(T-t))}{T-t}}. \tag{1.1}$$

Moreover the shape of the cusp that forms at time T in the corresponding family of curves is given by

$$y = \left(\frac{\pi}{4} + o(1)\right)\frac{x}{\ln \ln(1/x)}. \tag{1.2}$$

The hypotheses will be satisfied if the $\gamma(t)$ are cardioid-shaped curves, by which we mean any curve with winding number 2, which is invariant under reflection in the x-axis, and on which the curvature has one maximum, one minimum, and no other critical points in between these. In [A, §11] it is shown that if $\gamma(0)$ is cardioid-like then the entire evolution $\gamma(t)$ defined by $\gamma(0)$ is cardioid-like and will form a singularity to which Theorem 1.1 may be applied.

The formation of singularities of solutions of curve shortening has been studied by a number of authors [GH, Gr, Alt, A]. In particular in [A] it was shown that for any solution $\{\gamma(t): 0 < t < T\}$ of Curve Shortening satisfying our hypotheses the maximal curvature

$$M(t) = \max_{\gamma(t)} k$$

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