Vol. 77, No. 1 DUKE MATHEMATICAL JOURNAL © January 1995

CHARACTERIZATIONS OF CHARACTER SHEAVES FOR COMPLEX REDUCTIVE ALGEBRAIC GROUPS

E. GALAKTIONOVA

1. Introduction. Let G be a connected reductive algebraic group over an algebraically closed field. Character sheaves on G are certain G-equivariant irreducible perverse sheaves on G. They lead to a method of calculating irreducible characters of finite Chevalley groups. In [7, Chapter 13] G. Lusztig suggested a description of character sheaves for a complex group G. It was motivated by a work of Harish-Chandra [2] on characters of real and p-adic groups. This initial description is quite different from the definition of character sheaves given by Lusztig later in [9] for a group defined over a field of arbitrary characteristic. It is shown here, in particular, that the original description of character sheaves is equivalent to the later definition.

Harish-Chandra's work also motivated a characterization of character sheaves in terms of microsupport. This characterization was conjectured by Laumon [6] and Lusztig and proved independently by Ginsburg in [4] and by Mirković and Vilonen in [10]. Let us fix some notation. Let G be defined over the complex numbers. Let $\mathfrak{g}$ be its Lie algebra and $\mathcal{N}$ the nilpotent cone of $\mathfrak{g}$. Let $\mathcal{F}$ be a G-equivariant (for conjugation) perverse sheaf on G. $SS(\mathcal{F})$ denotes the microsupport of $\mathcal{F}$; $SS(\mathcal{F})|_S = SS(\mathcal{F}) \cap T^*G|_S$ is the restriction of the microsupport to the set S of all semisimple elements of G; $\mu_Y(\mathcal{F})$ is the microlocalization of $\mathcal{F}$ along a submanifold Y of G.

THEOREM 1.1. *The following are equivalent:*

(i) $SS(\mathcal{F}) \subseteq G \times \mathcal{N}$;

(ii) $SS(\mathcal{F})|_S \subseteq S \times \mathcal{N}$;

(iii) $\operatorname{supp} \mu_{\mathcal{O}}\mathcal{F} \subseteq \mathcal{O} \times \mathcal{N}$, *for all conjugacy classes* $\mathcal{O} \subset G$;

(iv) $\operatorname{supp} \mu_x\mathcal{F} \subseteq \mathcal{N}$, *for all* $x \in G$;

(v) $\operatorname{supp} \mu_{\mathcal{O}}\mathcal{F} \subseteq \mathcal{O} \times \mathcal{N}$, *for all conjugacy classes* $\mathcal{O} \subset S$, *and* $\mathcal{F}$ *is constructible for the Lusztig stratification.*

Ginsburg in [4] and Mirković and Vilonen in [10] proved that *character sheaves* can be equivalently defined as irreducible G-equivariant perverse sheaves on G satisfying (i). Thus (ii)–(v) could be used as well to define character sheaves. Characterizations of this kind essentially simplify the development and understanding of the theory. Currently there is no such description for groups over a field of positive characteristic. J. Bernstein conjectured that (v) characterizes character sheaves for groups in arbitrary characteristic.

Received 28 January 1994. Revision received 14 June 1994.