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ELLIPTIC DUNKL OPERATORS, ROOT SYSTEMS, AND FUNCTIONAL EQUATIONS

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Introduction. In the paper [1], Dunkl introduced the following difference-differential operators acting on functions on a Euclidean space V, related to arbitrary finite groups G generated by orthogonal reflections in V:

$$\nabla_\xi = \partial_\xi + \sum_{\alpha \in R_+} k_\alpha(\alpha, \xi)\frac{1}{(\alpha, x)}\hat{s}_\alpha. \tag{1}$$

Here ∂_ξ denotes the partial derivative in the direction $\xi \in V$, R is the root system of the group G, i.e., the set of unit normals to the reflection hyperplanes, R_+ is its positive part with respect to some generic linear form on V, $k_\alpha = k(\alpha)$ is a G-invariant function on R, s_α is the reflection corresponding to the root $\alpha \in R$, and $\hat{s}_\alpha$ is the operator on the space of functions on V:

$$\hat{s}_\alpha f(x) = f(s_\alpha(x)).$$

To be precise, Dunkl used slightly different operators, which are conjugated to (1) by the operator of multiplication by $\prod (\alpha, x)^{k_\alpha}$.

The main property of the Dunkl operators is given by the following.

THEOREM 1 (Dunkl). *The operators (1) commute with each other:*

$$[\nabla_\xi, \nabla_\eta] = 0, \tag{2}$$

for all $\xi, \eta \in V$.

The goal of this work is to describe certain generalizations of the Dunkl operators (1), preserving the property (2). Some of these results were announced in [2].

In Section 1 we consider generalizations of the form

$$\nabla_\xi = \partial_\xi + \sum_{\alpha \in A_+} k_\alpha(\alpha, \xi)\frac{1}{(\alpha, x)}\hat{s}_\alpha, \tag{3}$$

where A_+ is the set of unit normals to some set S of hyperplanes in V passing through the origin, A_+ is its positive part, and $k_\alpha = k(\alpha)$ is some function on A_+. We show that the commutativity of the operators ∇_ξ implies that S is the set of reflection hyperplanes of some Coxeter group G, $A_+ = R_+$, and k is G-invariant..

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