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THE HYPERELLIPTIC LOCUS

CRIS POOR

Introduction. The Schottky problem is the problem of characterizing Jacobians among all abelian varieties. In 1888, for genus four, Schottky gave a homogeneous polynomial in the theta constants which vanishes on $\mathscr{H}_4$ precisely at the Jacobian points; a proof of this was finally published by Igusa in 1981 [11]. A solution of the Schottky problem in general, such as the one given by Schottky and Igusa in genus four, would be a set of polynomials in the theta constants which vanish precisely on the Jacobian locus of $\mathscr{H}_g$, the Siegel upper half-space. These equations have proved elusive, whereas other interesting methods of characterizing Jacobians have met with more success; here however we restrict ourselves to the approach which requires the specification of a sufficient number of equations.

Along the same lines as the Schottky problem, we may consider other Schottky-type problems, such as the characterization of hyperelliptic Jacobians among all abelian varieties, the topic of this paper. It was known to Schottky by 1880 [12, page 763] that a Jacobian of genus three is hyperelliptic precisely when an even theta constant vanishes. Great progress was made in 1984 when Mumford, using the methods of dynamical systems, characterized hyperelliptic Jacobians among all abelian varieties by the vanishing and nonvanishing of certain theta constants [14]. For simplicity and strength this theorem can hardly be improved; from the point of view of the original Schottky problem, however, and from the desire to have an algebraic description of moduli space, it is beneficial to replace the nonvanishing conditions by further equalities. That the vanishing conditions define hyperelliptic Jacobians among all irreducible abelian varieties is proven in Theorem 2.6.1 and is a solution of the Schottky-type problem for hyperelliptic curves of arbitrary genus. One still wonders if the irreducibility hypothesis can be removed. The result is new for $g \geqslant 5$ and is encouraging because it is a result for each genus which does not demand the existence of auxiliary parameters or nonvanishing conditions.

MAIN THEOREM 2.6.1. *Let $\eta \in \Xi_g$ and $\Omega \in \mathscr{H}_g$. The following two statements are equivalent.*

(1) *Ω is irreducible and Ω satisfies the equations $V_{g,\eta}$.*

(2) *There is a marked hyperelliptic Riemann surface M of genus g which has Ω as its period matrix and* $\mathrm{Jac}(M) = \mathbb{C}^g/(\mathbb{Z}^g + \Omega\mathbb{Z}^g)$. *Furthermore, there is a model of M, $y^2 = \prod_{i \in B}(x - a_i)$, with a_∞ as the basepoint of the Abel-Jacobi map $w: M \to$* $\mathrm{Jac}(M)$ *such that $w(a_i) = [(\Omega I)\eta_i]$ in* $\mathrm{Jac}(M)$.

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