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A PIERI FORMULA IN THE GROTHENDIECK RING OF A FLAG BUNDLE

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1. Introduction. We first explain our result in geometric language. Let V be a vector bundle of rank n on a variety X, and let $Fl(V)$ be the bundle of complete flags in V, $\rho\colon Fl(V) \to X$ the projection, with $E_1 \subset E_2 \subset \cdots \subset E_n = \rho^* V$ the universal (tautological) flag of bundles on $Fl(V)$. Suppose we are given a complete flag $F_1 \subset \cdots \subset F_n = V$ of subbundles of V is given on X. Define line bundles L_i on $Fl(V)$ and M_i on X by

$$L_i = E_{n+1-i}/E_{n-i}, \qquad M_i = F_i/F_{i-1}, \qquad 1 \leqslant i \leqslant n.$$

For each partition $\lambda = (\lambda_1 \geqslant \cdots \geqslant \lambda_n \geqslant 0)$, let $L^\lambda = L_1^{\otimes \lambda_1} \otimes \cdots \otimes L_n^{\otimes \lambda_n}$.

For each permutation w in the symmetric group S_n, there is a *Schubert variety* $\Omega_w \subset Fl(V)$, of codimension the length $\ell(w)$ of the permutation; Ω_w is the locus where, for all p and q, the canonical maps from $\rho^* F_p$ to $\rho^*(V)/E_{n-q}$ have ranks bounded by numbers determined by w:

$$\operatorname{rank}(\rho^* F_p \to \rho^*(V)/E_{n-q}) \leqslant \operatorname{Card}\{i \leqslant q\colon w(i) \leqslant p\}.$$

Since the structure sheaves of these Schubert varieties have finite resolutions by vector bundles (see Remark 5 below), they determine classes $[\mathcal{O}_{\Omega_w}]$ in the Grothendieck ring $K^0 Fl(V)$ of vector bundles on $Fl(V)$, and these classes form a basis for $K^0 Fl(V)$ as a module over $K^0 X$, as w varies over the symmetric group. The goal of this article is to describe the class in $K^0 F\ (V)l$of the restriction of any bundle L^λ to any Schubert variety Ω_w in terms of this basis. Our answer is a formula

$$[L^\lambda|_{\Omega_w}] = \sum [M^T] \cdot [\mathcal{O}_{\Omega_{v(T,w)}}], \tag{1}$$

where the sum is over a certain set of tableaux T of shape λ with entries in $\{1, \ldots, n\}$.[1] The bundle M^T is the tensor product $\bigotimes_{i=1}^n M_i^{\otimes m(i)}$, where $m(i)$ is the number of times i occurs in the tableau T. The set of tableaux in the sum and the definition of the permutation $v(T, w)$ require some concepts from the calculus of tableaux and will be described later. It is striking that the coefficients of the structure sheaves are all positive.

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[1] A *tableau* of shape λ is a numbering of the boxes of the Young diagram of λ with numbers from $\{1, \ldots, n\}$, which are weakly increasing in the rows, and strictly increasing in the columns (read down in English, up in French). See [18], [9], [20], [4].