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FURTHER IMPROVEMENTS IN WARING'S PROBLEM, II: SIXTH POWERS

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1. Introduction. In recent years there has been a series of developments in the theory of Waring's problem, following the introduction of the use of numbers with only small prime factors in Vaughan [3]. This has occurred through the provision of upper bounds for the number of solutions, $S_s^{(k)}(P, R)$, of the diophantine equations

$$x_1^k + \cdots + x_s^k = y_1^k + \cdots + y_s^k,$$

with $x_i, y_i \in \mathscr{A}(P, R)$ $(1 \leqslant i \leqslant s)$, where throughout we write

$$\mathscr{A}(P, R) = \{n \in \mathbb{Z} \cap [1, P]: p \text{ prime}, p|n \text{ implies } p \leqslant R\}.$$

When $R = P^\eta$, with $\eta = \eta(\varepsilon, s, k)$ a sufficiently small but fixed positive number, such bounds take the form

$$S_s^{(k)}(P, R) \ll P^{\lambda_s + \varepsilon}.$$

As a consequence of further developments due to Wooley [6], this has led in Vaughan and Wooley [5] to the upper bounds

$$G(5) \leqslant 17, \qquad G(6) \leqslant 25, \qquad G(7) \leqslant 33, \qquad G(8) \leqslant 43, \qquad G(9) \leqslant 51,$$

where, as usual, we write $G(k)$ for the smallest number s such that every sufficiently large natural number s is the sum of, at most, s kth powers of natural numbers. In the case of $k = 6$, we were able in [5] to prove that

$$S_{12}^{(6)}(P, R) \ll P^{18+\varepsilon}. \tag{1.1}$$

An expert in the Hardy-Littlewood method might, at first sight, expect that a further refinement would lead relatively easily to $G(6) \leqslant 24$. However, a perusal

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