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GEODESICS OF HOFER'S METRIC ON THE GROUP OF HAMILTONIAN DIFFEOMORPHISMS

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§1. Introduction and main results. In the present paper, we study geometry of the group $\mathscr{D}$ of compactly supported Hamiltonian diffeomorphisms of $\mathbb{R}^{2n}$ endowed with Hofer's metric (see [H1], [H2], [H-Z]). Our basic observation is that each point of this group has a *flat* C^1-neighborhood (see 1.2). This allows us to give a complete description of *geodesics* on $\mathscr{D}$ (see 1.3).

Our approach is based on variational methods developed in [H1], [H2], [HZ]. We also present an application of these results to classical mechanics. Namely, we discuss interrelations between invariant tori of optical Hamiltonian flows on $T^*\mathbb{T}^n$ and their metrical properties (see 1.4).

1.1. Preliminaries. Consider the standard linear symplectic space $(\mathbb{R}^{2n}, \omega)$. A *smooth path* of symplectomorphisms of $\mathbb{R}^{2n}$ is an isotopy generated by a smooth compactly supported Hamiltonian function. Let $\mathscr{D}$ be the (infinite-dimensional) Lie group of all symplectomorphisms of $\mathbb{R}^{2n}$ which can be joined with the identity map by a smooth path. We identify the Lie algebra $\mathfrak{d}$ of $\mathscr{D}$ with $C_0^\infty(\mathbb{R}^{2n})$.

Let $\| \ \|$ be a norm on $\mathfrak{d}$, $\|H\| = \max H - \min H$. Since this norm is invariant under adjoint action of $\mathscr{D}$, it defines a bi-invariant Finsler metric on $\mathscr{D}$, and hence, in the standard way, a length structure and a (pseudo)-distance. Namely, given a smooth path

$$\ell\colon [a, b] \to \mathscr{D},$$

we set $\text{length}(\ell) = \int_a^b \|\dot{l}(t)\|\, dt$, and for two elements $\varphi, \psi \in \mathscr{D}$ we define

$$d(\varphi, \psi) = \inf \text{length}(\ell),$$

where the infimum is taken over all smooth paths ℓ on $\mathscr{D}$ joining φ and ψ. A nontrivial result by H. Hofer [H1] states that d is a genuine distance function on $\mathscr{D}$.

1.2. C^1-Flatness. By definition, Hofer's distance on $\mathscr{D}$ is introduced via length of smooth paths. These paths are continuous in C^1-Whitney topology which is *finer* than one induced by the distance. Therefore, it is important to understand geometry of C^1-Whitney open sets on $\mathscr{D}$. For this purpose, we use a classical tool

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