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BROWNIAN MOTION AND THE FUNDAMENTAL FREQUENCY OF A DRUM

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0. Introduction. In 1976, W. K. Hayman [11] solved an important problem in the study of vibrating membranes raised by Polya and Szegö in their famous book *Isoperimetric Inequalities in Mathematical Physics* [23, page 16], by proving the following.

THEOREM (Hayman). *Let D be a simply connected domain in the complex plane. Let R_D be the inradius of D, that is, the radius of the largest disc contained in D, and let λ_D be the first Dirichlet eigenvalue for the Laplacian in D. There is a universal constant a such that*

$$\lambda_D \geqslant \frac{a}{R_D^2}. \tag{0.1}$$

Thus, if a drum produces an arbitrarily low tone, it must necessarily contain an arbitrarily large circular drum. The opposite inequality is trivial. If U denotes the disc of radius R_D, then $\lambda_U = j_0^2/R_D^2$, where $j_0 \approx 2.4048$, is the smallest positive zero of the first Bessel function J_0. By the domain monotonicity of the eigenvalue, $\lambda_D \leqslant \lambda_U = j_0^2/R_D^2$.

There have been many efforts to find the best constant a and to identify the extremal domain in the inequality (0.1). Hayman's original proof gave $a = 1/900$. The references [1], [2], [6], [7], [10], [20], and [27] contain various proofs and extensions of Hayman's theorem with various values of a. The best known value for a is 1/4 given by R. Osserman [20]; see also C. Croke [7]. It was proved by J. Hersch [13] that $\lambda_D \geqslant \pi^2/(4R_D^2)$ for convex D, with equality if and only if D is an infinite strip. This had been conjectured by Polya and Szegö [23, page 17]. In [20], Osserman suggested that $a \geqslant \pi^2/4$ might also hold for arbitrary simply connected domains. However, as pointed out by the referee of Osserman's paper, this is not the case. Osserman then stated [20, page 554] that 1/4 "may be a candidate for the best constant" in (0.1). In this paper we prove that $a > 0.6197$ and provide examples of domains which we believe are close to extremal for the inequality (0.1). We do this by relating the above problem to two other extremal problems, one concerning the expected lifetime of Brownian motion in D and the other concerning the density of the hyperbolic metric in D. We find this connection to

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