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RATIONAL POINTS ON NONSINGULAR CUBIC HYPERSURFACES

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1. Introduction. Let $X \subset \mathbb{P}^n_{\mathbb{Q}}$ be a projective hypersurface of degree m defined over the rational numbers $\mathbb{Q}$. The study of the existence of $\mathbb{Q}$-rational points on X has a long history. A well-known theorem of Meyer states that if $m = 2$ and $n \geqslant 4$, then $X(\mathbb{Q}) \neq \emptyset$ provided that X is defined by an indefinite quadratic form. For hypersurfaces of higher degree, the first substantial result was that of Tartakowski [T], who proved that a "general" cubic hypersurface has a $\mathbb{Q}$-rational point if n is sufficiently large. Subsequently, Lewis [L] proved a sufficient condition that is applicable to any cubic hypersurface. Almost simultaneously, similar results were proved by Birch [B1] and Davenport [D1]. Birch's work, in fact, gave a sufficient condition for the existence of a $\mathbb{Q}$-rational point on a hypersurface of any odd degree, while Davenport markedly reduced the lower bound on n required of a cubic hypersurface from around 1000 to 31. The methods of Lewis and Birch involve "splitting off cubes" (or higher powers in Birch's case), reducing the problem to that for a diagonal hypersurface of smaller dimension and then applying the Hardy-Littlewood method to a diagonal equation defining the hypersurface, obtaining an upper bound on the dimension needed. This method requires n to be very large. Davenport's reduction, and its subsequent improvements in [D2] and [D3] culminating with the bound of 15, were the result of the successful application of the Hardy-Littlewood method to a cubic form defining the original hypersurface (though in [D1] this is intertwined with the idea of splitting off cubes). The method of Davenport was further developed by Birch [B2] and, more recently, by Schmidt [S], both of whom establish conditions for the existence of $\mathbb{Q}$-rational points on the intersection of hypersurfaces. In Birch's work, the hypersurfaces are required to be of the same degree, and a condition is imposed on the dimension of the locus of complex singularities. Schmidt's work applies to systems of hypersurfaces of varying degrees. Recently, for certain restricted classes of cubic hypersurfaces defined over $\mathbb{Q}$, Heath-Brown [HB] and Hooley [H1], [H2] have reduced Davenport's bound of 15. Heath-Brown proved that if $m = 3$, $n \geqslant 9$, and X is nonsingular, then $X(\mathbb{Q}) \neq \emptyset$. In [H1], Hooley replaced $n \geqslant 9$ with $n \geqslant 8$ under the additional assumption that $X(\mathbb{Q}_p) \neq \emptyset$ for all primes p. In [H2], he replaced nonsingularity with the weaker condition that the singular locus consist of isolated double points, linearly independent over $\overline{\mathbb{Q}}$, the algebraic closure of $\mathbb{Q}$. The results in [HB], [H1], and [H2] are obtained via some delicate modifications of the methods

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