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EVOLVING PLANE CURVES BY CURVATURE IN RELATIVE GEOMETRIES II

MICHAEL E. GAGE AND YI LI

0. Introduction. In this paper we prove the existence of self-similar solutions to the anisotropic curve shortening equation.

THEOREM 0.1. *Given any positive C^2 function γ on S^1 there exists a solution to the equation*

$$\frac{\partial X}{\partial t} = \gamma(\theta)kN, \tag{0.1}$$

which is self-similar. This means that the evolution shrinks the initial curve without changing its shape.

In (0.1) $X: S^1 \times [0, \omega) \to \mathbb{R}^2$ is the position vector of a family of closed convex plane curves, kN is the curvature vector, with k being the curvature and N the inward pointing normal given by $N = -(\cos\theta, \sin\theta)$. The weight function $\gamma(\theta) = \gamma(N)$ is a function of the normal vector to the curve at each point, but does not depend on position in the plane.

Equation (0.1) has two significant interpretations. It can be seen as the generalization of the "curve shortening" problem [Ga3] to Minkowski geometry or as a simplified model of the motion of the interface of a metal crystal as it melts [AnGu], [Ta1], [Ga3].

The proofs given in this paper illustrate most of the current techniques that are being used to understand geometric evolution equations as described in, for example, [Ha3].

It is not hard to show that the self-similar solutions correspond to positive, 2π periodic solutions of the equation

$$\frac{d^2h}{d\theta^2} + h = \frac{\gamma(\theta)}{h}, \tag{0.2}$$

and that $h(\theta)$ is the support function of the suitably normalized self-similar solution. (This follows from equation (1.1) and either equation (2.1) or (5.2)).

This allows us to reinterpret theorem (0.1) as

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