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# ROOT MULTIPLICITIES OF KAC-MOODY ALGEBRAS

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*To Professor George B. Seligman on his 65th birthday*

**Introduction.** In [K1] and [M], Kac and Moody independently introduced a new class of Lie algebras, called *Kac-Moody algebras*, associated with generalized Cartan matrices. In [Ma], by generalizing Weyl's denominator identity to the affine root system, Macdonald obtained a new family of combinatorial identities, including Jacobi's triple product identity as the simplest case. In [K2], Kac discovered a character formula, called the *Weyl-Kac formula*, for integrable highest-weight modules over symmetrizable Kac-Moody algebras, which generalizes Weyl's character formula for finite-dimensional irreducible modules over finite-dimensional simple Lie algebras. The Weyl-Kac formula, when applied to the 1-dimensional trivial representation, yields the *denominator identity*, and the Macdonald identities are equivalent to the denominator identities for affine Kac-Moody algebras.

The Kac-Moody algebras are usually infinite dimensional, and have a decomposition into a direct sum of finite-dimensional subspaces called the *root spaces*. The dimension of a root space is called the *multiplicity* of the root attached to the root space. In [BM], using the denominator identity, Berman and Moody derived a closed form formula for the root multiplicities of all symmetrizable Kac-Moody algebras. It is the first formula for the root multiplicities of Kac-Moody algebras and turns out to be very effective for the rank 2 Kac-Moody algebras (see Section 6). As another application of the denominator identity, Peterson obtained a Freudenthal-type recursive formula for the root multiplicities [P].

In this paper, using the Euler-Poincaré principle and Kostant's formula, we obtain two new root multiplicity formulas for all symmetrizable Kac-Moody algebras: one is in recursive form, and the other is in closed form. These formulas enable us to study the structure of a symmetrizable Kac-Moody algebra $\mathfrak{g}$ as a representation of the smaller Kac-Moody algebra $\mathfrak{g}_0$ contained in $\mathfrak{g}$. To be more precise, if we have a $\mathbf{Z}$-gradation $\mathfrak{g} = \bigoplus_{j \in \mathbf{Z}} \mathfrak{g}_j$ on a symmetrizable Kac-Moody algebra $\mathfrak{g}$ such that the subalgebra $\mathfrak{g}_0$ is a Kac-Moody algebra with an extended Cartan subalgebra, then all the homogeneous subspaces $\mathfrak{g}_j$ ($j \neq 0$) are integrable representations of $\mathfrak{g}_0$, and hence we can use the representation theory of $\mathfrak{g}_0$. Let $\mathfrak{g}_\pm = \bigoplus_{j \geq 1} \mathfrak{g}_{\pm j}$. Since the structure of the algebra $\mathfrak{g}$ is symmetric, we concentrate on the study of the negative part $\mathfrak{g}_- = \bigoplus_{j \geq 1} \mathfrak{g}_{-j}$.

We believe that the new recursive formula can be used to determine the $\mathfrak{g}_0$-module

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