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# CONFORMAL IMMERSIONS OF COMPLETE RIEMANNIAN MANIFOLDS AND EXTENSIONS OF THE SCHWARZ LEMMA

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**Introduction.** The main aim of this article is to study conformal immersions $\phi: (M, g) \to (N, h)$. We always assume that $(M, g)$ is a complete, connected, noncompact Riemannian manifold of dimension $m$. To simplify the exposition, we suppose that $m \geqslant 3$ until Section 1 below. The scalar curvature of $(M, g)$ is denoted by $s(x)$, while the scalar curvature of the pull-back metric $\phi^*h$ is called $K(x)$. The conformality factor of $\phi$ is determined by writing $\phi^*h = u^{4/(m-2)}g$, with $u(x) > 0$ for all $x \in M$. The map $\phi$ is said to be *weakly distance decreasing* if $u \leqslant 1$ on $M$.

We work along lines initiated, in the compact case, by Lichnerowicz [10], Obata [11], and Yano and Nagano [14]; and developed, in the complete case, by Yau [15]: in particular, we seek conditions on the curvature of $(M, g)$ which ensure that any conformal immersion $\phi$ with $K(x) \leqslant s(x)$ is weakly distance decreasing. Our main result is the following.

THEOREM 1. *Let $r(x)$ be the distance function from a fixed point $p \in (M, g)$. Suppose that*

(0.1) $\mathrm{Ricc}_{(M,g)} \geqslant -(m-1)B^2(1 + r(x))^{2(1-\gamma)}$ *for some $B > 0$ and $\gamma \leqslant 2$;*

*Let $\phi: (M, g) \to (N, h)$ be a conformal immersion such that*

(0.2) $K(x) \leqslant \min\{0, s(x)\}$ *for all $x \in M$;*

(0.3) $K(x) \leqslant -\dfrac{d^2}{(1 + r(x))^\gamma}$ *if $r(x) \gg 1$, for some $d > 0$ ($\gamma$ as in (0.1)).*

*Then $\phi$ is weakly distance decreasing.*

An immediate consequence of Theorem 1 is the following.

COROLLARY 1. *Assume that (0.1) holds on $(M, g)$. Suppose furthermore that the scalar curvature $s(x)$ of $(M, g)$ is nonpositive and satisfies*

(0.4) $s(x) \leqslant -\dfrac{d^2}{(1 + r(x))^\gamma}$ *if $r(x) \gg 1$, for some $d > 0$ ($\gamma$ as in (0.1)).*

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