PERIODICITY OF THE FIXED LOCUS OF MULTIPLES
OF A DIVISOR ON A SURFACE

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In his paper "The theorem of Riemann-Roch for high multiples of an effective
divisor on an algebraic surface" [Z], Zariski studies the Riemann-Roch problem
on an algebraic surface. His main results are presented in the final section of his
paper, titled "A Summary of Principal Results".

Let D be an effective divisor on a nonsingular projective surface S over an
algebraically closed field k. Zariski shows that dim \text{k} nD is a quadratic polynomial
in n plus a bounded function of n. Zariski shows that this bounded function is
eventually periodic in some cases, and he asks if the function is eventually periodic
for an arbitrary effective divisor on S.

This question is answered in [CS]. We give an affirmative answer to this question
when k has characteristic zero [CS, Thm. 2], and we provide a counterexample in
positive characteristic [CS, Ex. 3].

Zariski also considers the behaviour of the fixed locus of |nD| for large n. The
fixed component of D is the largest effective divisor E such that L \cdot E is effective
for all L in the complete linear system |nD|. Let B_n be the fixed component of |nD|.
Zariski shows that there exists a divisor E on S such that B_n \cdot nE is bounded. Zariski
proves that B_n \cdot nE is eventually periodic in some cases. It is natural to ask if this
divisor is eventually periodic for an arbitrary effective divisor on S.

In this paper (Theorem 5) we give a positive answer to this question. We prove
that the divisor B_n \cdot nE is periodic in n for an arbitrary effective divisor D on a
surface S if k has characteristic zero. However, [CS, Ex. 3] shows that periodicity
of the fixed component does not hold in positive characteristic.

An example is given in Section 3 of this paper showing that the codimension-two
part of the base locus can be nonperiodic, even on a surface in characteristic zero.

An example showing nonpolynomial-like growth of the fixed locus for divisors
on a three-fold is given in [CS, §7].

Our main technical tool in proving Theorem 5 is [CS, Thm. 8], a periodicity
theorem on cohomology. This is stated in Theorem 4 in Section 1 of this paper.

We will use the following notation. A Cartier divisor D (or a line bundle \mathcal{L})
on a nonsingular projective variety X is nef if D (respectively \mathcal{L}) has a non-
negative intersection number with every curve on X. For any proper k-scheme
X, a line bundle \mathcal{L} on X is numerically trivial if its restriction to every integral
curve in X has degree 0. A \mathbb{Q}-divisor is a rational sum of integral (prime) divisors.