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A SHARP COUNTEREXAMPLE TO THE LOCAL EXISTENCE OF LOW-REGULARITY SOLUTIONS TO NONLINEAR WAVE EQUATIONS

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0. Introduction. Recently, starting with Klainerman-Machedon [11], there has been a lot of activity in minimizing the regularity assumptions necessary to ensure local existence and uniqueness for systems of nonlinear wave equations. Other more recent articles are by Ponce-Sideris [14] and Beals-Bezard [1]. However, we have lacked nontrivial counterexamples; the gap between what the trivial scaling counterexamples give and what is suggested by the existence result is 1/2 derivative in the Sobolev norm of data. But, as we shall see below, it is simple to find counterexamples reducing this to an arbitrarily small gap. For the simplest model equation we have even found a sharp counterexample. For the proof of this, we use techniques developed in Lindblad [12].

We are concerned with the question what the smallest s is such that we have local existence and uniqueness in H_s of the problem

$$\tag{0.1} \begin{gathered} \Box u = G(u, u'), \qquad 0 \leqslant t < T \\ u(0, x) = u_0(x) \in H_s(\mathbf{R}^3), \qquad \partial_t u(0, x) = u_1(x) \in H_{s-1}(\mathbf{R}^3), \end{gathered}$$

where $G(u, u')$ is a smooth function of u and $u' = (u_t, \nabla_x u)$. (Here $\Box = \partial_t^2 - \sum_{i=1}^3 \partial_{x_i}^2$.) In other words we want to know if we have a unique distributional solution of (0.1) for some $T > 0$ with the norm

$$\tag{0.2} \|u(t, \cdot)\|_s^2 = \int (||D_x|^{s-1} u_t(t, x)|^2 + ||D_x|^s u(t, x)|^2)\, dx$$

bounded for $0 \leqslant t < T$, provided that this norm is bounded initially. (Here $|D_x| = (-\Delta_x)^{1/2}$.) It is natural to ask for the solution to be strongly continuous in H_s; $\|u(t, \cdot) - u(\tau, \cdot)\|_s \to 0$ when $t \to \tau < T$, so that the initial data problem is well defined in H_s. It is also natural to ask for the problem to be well posed in H_s for $t < T$, which usually means that, for every sequence of solutions $\{u_k\}$ with $\|u_j(0, \cdot) - u_k(0, \cdot)\|_s \to 0$ as $j, k \to \infty$, we have that $\|u_j(t, \cdot) - u_k(t, \cdot)\|_s \to 0$ as $j, k \to \infty$, for $t < T$. (We are, however, only interested in the local question; so we will only ask for this when initial data for all the functions u_k, $k = 1, 2, \ldots$, are supported in a fixed

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