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COMPACTNESS OF CONFORMAL METRICS WITH INTEGRAL BOUNDS ON CURVATURE

MATTHEW J. GURSKY

Introduction. The problem we will consider here is from conformal geometry and can be roughly stated as follows. Suppose that (M, g_0) is a compact Riemannian manifold and that we conformally deform g_0 to obtain a new metric g, i.e., that $g = e^f g_0$ where f is a smooth function. Suppose further that we have a priori geometric information about g (e.g., bounds on curvature and volume); can we recover some quantitative information about the conformal factor f (e.g., pointwise bounds or regularity properties)? Before we formulate this question in a more precise manner, let us consider some background material and examples of where it naturally arises.

In what may be thought of as an attempt to generalize the uniformization theorem to higher dimensions, Yamabe [Y] asked whether a given compact n-dimensional manifold ($n \geqslant 3$) is conformal to one of constant scalar curvature. His approach was to attempt to solve the elliptic equation satisfied by a conformal metric of constant scalar curvature using the methods of the calculus of variations. Although he claimed to have answered the question in the affirmative, Trudinger [T] found an error in Yamabe's proof which he was only able to fix if a certain conformal invariant, called the Yamabe invariant, $Q(M, g_0)$ is less than 0. (We will have more to say about the Yamabe invariant later. For now it suffices to say that there is an analogy between the role played by $Q(M, g_0)$ in the solution of the Yamabe problem and the role played by the Euler characteristic of a compact surface in the uniformization theorem.) The complete solution of the Yamabe problem, that is, the treatment of the case where $Q(M, g_0) > 0$, took over sixteen years (for an exhaustive survey see [LP]).

The difficulty of the problem lies in the conformal invariance of the equation to be solved and the fact that the conformal group of the standard sphere is noncompact. Therefore, a sequence of conformal metrics with fixed volume and scalar curvature approaching a constant may not converge. The key to solving the problem was finding a way to distinguish whether a given manifold was conformally equivalent to the standard sphere. The proof of the positive mass conjecture by Schoen and Yau provided such a criterion, and Schoen [S1], [S2], and [SY] used it to complete the solution of the Yamabe problem.

There have been similar attempts to prove the uniformization theorem by variational techniques. One such program was carried out by Osgood, Phillips, and Sarnak [OPS1], who showed that the "uniform" metrics of constant curvature

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