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EXTREMAL PROBLEMS INVOLVING LOGARITHMIC AND GREEN CAPACITY

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1. Introduction. How small can the logarithmic capacity of a connected set K be, if it has fixed Green capacity (i.e. hyperbolic capacity) in the unit disc and contains the origin? Since the Green capacity is a number between 0 and 1, we can think of it as the "proportion" of the hyperbolic disc occupied by K. Similarly, the logarithmic capacity measures how much of the plane K covers. We will give precise definitions later. The goal, then, is to make K occupy as little as possible of the plane given that it occupies a fixed proportion of the hyperbolic disc. The requirement that K contain the origin prevents us from simply applying Möbius transformations to push K out to the boundary of the disc, thereby making its logarithmic capacity arbitrarily small.

V. V. Koževnikov raised this question orally at a seminar in 1980, then D. Gaier independently raised it in writing in joint work [10] with W. K. Hayman on the computation of modules of thick ring domains. Gaier's conjecture was that the logarithmic capacity is minimal when K is the line segment $[0, b]$, with b chosen to give the desired value for the Green capacity. Gaier further suggested in [17, p. 112] that this capacity conjecture should also be true when the ambient domain (with respect to which the Green capacity of K is calculated) is the halfplane $\{\operatorname{Re} z < 1\}$ rather than the unit disc. R. Kühnau [17, Satz 1] made the first progress towards proving these conjectures when he used interior and boundary variational techniques to prove the capacity conjecture in the "disc" case provided $0 < b < 0.276$.

In this paper we use potential-theoretic methods and A. Baernstein's $*$-function symmetrization technique to completely prove a strong and sharp generalization, Theorem 7, of the capacity conjecture for a class of ambient domains that includes the unit disc $\{|z| < 1\}$, the halfplane $\{\operatorname{Re} z < 1\}$, the slit plane $\mathbf{C}\backslash[\frac{1}{4}, \infty)$, the cardioid $\{z - \frac{1}{2}z^2 : |z| < 1\}$, and all nth roots of these domains. A. Yu. Solynin [23] has independently proved the "disc" and "halfplane" cases of the capacity conjecture, using Hadamard's variation and the Golusin-Komatu equation. Since the Hadamard variation requires some normal displacement of a smooth boundary, it is unclear whether Solynin's method can be applied when the boundary of the ambient domain is not smooth or when normal displacement breaks down, as it does for $\mathbf{C}\backslash[\frac{1}{4}, \infty)$. Moreover, the Golusin-Komatu method appears to require that

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