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ON THE RADIAL VARIATION OF BOUNDED ANALYTIC FUNCTIONS ON THE DISC

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0. Introduction and statement of results. Let F be a bounded analytic function on the disc $D = \{z \in \mathbb{C} \,|\, |z| < 1\}$. Following [Ru], define

$$W(F, r, \theta) = \int_0^r |F'(\rho e^{i\theta})| d\rho (r < 1); \qquad V(F, \theta) = W(F, 1, \theta).$$

The quantity $V(F, \theta)$ naturally corresponds to the variation of F on the radius of D terminating at the point $e^{i\theta}$. It is shown in [R] that the set

$$\{\theta \,|\, V(F, \theta) < \infty\}$$

may be of measure zero (and of first category). In fact, as shown in [R], F may be taken to be a Blaschke product or an element of the disc algebra (i.e., continuous up to the boundary). The problem left open in [R] is whether $V(F, \theta)$ may be infinite in *any* direction. The purpose of this note is to disprove this. More precisely, we have the following result.

THEOREM 1. *The set $\{\theta \,|\, V(F, \theta) < \infty\}$ is nonempty and in fact is of Hausdorff dimension* 1, *whenever F is bounded analytic, i.e., $F \in H^\infty(D)$.*[1]

At this point, let us recall Zygmund's result [Zyg]

$$W(F, r, \theta) = 0\left(\log^{1/2} \frac{1}{1-r}\right), \qquad r \to 1,$$

almost everywhere in θ. This statement is optimal, also assuming $F \in H^\infty(D)$.

Our method also yields the following theorem.

THEOREM 2. *The statement of Theorem* 1 *holds assuming F a bounded real harmonic function on D.*

Theorem 2 has a martingale counterpart for real bounded dyadic martingales, obtained by taking conditional expectations with respect to the natural filtration on the Cantor group $G = \{1, -1\}^{\mathbb{N}}$. Writing

$$F(\varepsilon) = \sum_{k=1}^{\infty} \Delta_k F(\varepsilon_1, \ldots, \varepsilon_{k-1}) \varepsilon_k, \qquad \varepsilon = (\varepsilon_1, \varepsilon_2, \ldots) \in G,$$

[1] Also $J \cap \{\theta \,|\, v(F, \theta) < \infty\}$ is of Hausdorff dimension 1 for any arc J in $\mathbb{T}$.

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