Vol. 69, No. 3 DUKE MATHEMATICAL JOURNAL © March 1993

JORGENSEN'S INEQUALITY FOR DISCRETE GROUPS IN NORMED ALGEBRAS

SHMUEL FRIEDLAND AND SA'AR HERSONSKY

0. Introduction. Let $\mathscr{A}$ be a normed algebra with identity over **C** with the norm $|\cdot|$. Let $\Gamma \subset \mathscr{A}$ be a group of invertible elements in $\mathscr{A}$. We then regard Γ as a topological group with the topology induced by the norm $|\cdot|$. Assume that $a, b \in \mathscr{A}$ are invertible. Denote by $\langle a, b \rangle$ the group generated by a and b. Suppose that $\langle a, b \rangle$ is a discrete group. What can one say about a, b? The best known result in this area is Jorgensen's inequality [Jor]. Let $\mathscr{A} = M_2(\mathbf{C})$ be the algebra on 2×2 complex valued matrices and assume $\langle a, b \rangle$ is a subgroup of the special linear group of $SL_2(\mathbf{C}) \subset M_2(\mathbf{C})$. Then the sharp inequality of Jorgensen claims that if $\langle a, b \rangle$ do not generate an elementary group then

$$|(trace(a))^2 - 4| + |trace([a, b]) - 2| \geqslant 1. \tag{0.1}$$

Here, $[a, b] = aba^{-1}b^{-1}$. We call $\langle a, b \rangle$ an elementary group if and only if $\langle a, b \rangle$ has a nilpotent subgroup of a finite index. Note that (0.1) is invariant with respect to conjugacy in $GL_2(\mathbf{C})$. Jorgensen's inequality translates immediately to Kleinian groups—discrete groups of Möbius transformations of the Riemann sphere. Jorgensen's inequality was recently generalized by Martin [Mar1] to nonelementary discrete groups of Möbius transformations of any dimension $n > 2$. A simple version of Martin's inequality can be stated in our terms as follows. If $\langle a, b \rangle \subset \mathscr{A}$ generate a discrete nonelementary group then

$$\max(|a - 1|, |b - 1|) \geqslant 2 - \sqrt{3}. \tag{0.2}$$

To obtain the corresponding results for Möbius transformations one recalls that orientation preserving Möbius transformations are isomorphic to the group $SO^+(1, n)(\mathbf{R}) \subset M_{n+1}(\mathbf{C})$. In that case, the norm $|\cdot|$ is assumed to be the spectral norm. In [Mar2] Martin's inequalities are used to obtain new lower bounds for the volume of all hyperbolic n-manifolds.

The object of this paper is twofold. We first study necessary conditions for discreteness of the group $\langle a, b \rangle \subset \mathscr{A}$. By doing that we believe that one can get similar results for discrete groups acting on other homogenous spaces, for example the Siegel upper half plane. We also show that there are other variants of Martin's inequality (0.2). Second, we claim that if a, b belong to some classical groups, for example $SO^+(1, n)(\mathbf{R})$ or $Sp(n, \mathbf{R})$ then the inequality (0.2) can be improved by

Received 17 July 1992. Revision received 10 September 1992.