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# THE GEOMETRY AND TOPOLOGY OF QUOTIENT VARIETIES OF TORUS ACTIONS

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Geometric invariant theory assigns projective "quotient" varieties to any linear action of a complex reductive algebraic group $G$ on a projective variety $X$. In this paper, we restrict our attention to the case when $G$ is a complex torus $(\mathbb{C}^*)^n$ and study the geometry and topology of the quotient varieties of this $(\mathbb{C}^*)^n$ action.

As a main result of our program, we obtained a fairly explicit inductive formula for the intersection Betti numbers of an arbitrary quotient variety (see Sections 2.5 and 2.6). The formula involves certain isotropy subgroups of $(\mathbb{C}^*)^n$ and the fixed-point sets of these isotropy subgroups. But it does not involve the homology of $X$ nor a power series. Our method and formula are different from Kirwan's ([Ki]) in essence. In order to apply Kirwan's formula to singular quotients, one has to spend considerable effort on the desingularization of the singularities. However, from the viewpoint of this paper, in the case of torus actions, the procedure of designularization is not necessary. For one of our main results asserts that, whenever we have a singular quotient, we can always find a rationally nonsingular quotient and a canonical algebraic map from the latter to the former such that this map is rationally a *small resolution* in the sense of Goresky and MacPherson ([GM2]). An immediate consequence of this result is that, over the field of rational numbers, the homology groups of the rationally nonsingular quotients cover the intersection homology groups of all quotients (singular or not). However, we would like to observe that our homological formulae do not depend on the existence of small resolutions. The main tool leading to the formulae is the decomposition theorem of Beilinson-Bernstein-Deligne ([BBD]).

Beyond this, given a quotient, using the theorems in this paper, one will be able to tell rather explicitly what this quotient is by a finite number of simple inductive steps involving only fiber bundles, "nice" blowups and blowdowns. This provides information on both the intersection homology and the geometry. For instance, consider the moduli space $(\mathbb{P}_1^m)^{ss}//PGL(2)$ of semistable $m$ ordered points in a projective line up to projective equivalence. Using the Gelfand-MacPherson correspondence, we can identify it with a quotient in the Grassmannian $G(2, \mathbb{C}^m)$ acted on by $(\mathbb{C}^*)^{m-1}$. Then applying our method, we can describe explicitly how to obtain $(\mathbb{P}_1^m)^{ss}//PGL(2)$ from $\mathbb{P}^{m-3}$ by some explicit simple blowups and blowdowns. (This will be treated elsewhere).

The most important quotients under consideration in this paper are, of course, *symplectic quotients*, the quotients that can be identified with *symplectic reductions*,

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