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SYMPLECTIC STRUCTURES ON T^2-BUNDLES OVER T^2

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1. Introduction. A symplectic structure on a $2n$-dimensional differentiable manifold M is a closed, nondegenerate 2-form ω. The form ω being nondegenerate means that ω^n is nowhere zero (i.e., a volume form).

Associated with a symplectic form are the following topological data:

(i) a cohomology class $a = [\omega] \in H^2(M; \mathbb{R})$ with $a^n \neq 0$,

(ii) an almost complex structure J defined (up to homotopy) by the compatibility condition

$$\omega(JX, JY) = \omega(X, Y),$$

$$\omega(X, JX) > 0 \qquad (X \neq 0),$$

for all vector fields X, Y on M.

The general existence problem for symplectic structures is to decide whether every pair (a, J) on M is induced by a symplectic form ω. There is still no known example for which the answer to this question is negative, but it is now commonly believed that the answer will turn out to be negative in general.

Slightly easier to handle is the more restricted question of which cohomology classes a can be realized by a symplectic form. In this paper we investigate (smooth, orientable) T^2-bundles over T^2, where T^2 denotes the 2-dimensional torus, and show that every cohomology class $a \in H^2(M; \mathbb{R})$ with $a^2 \neq 0$ (M being the total space of the bundle) can be realized by a symplectic form.

This may not seem surprising since we are dealing with a bundle that has symplectic fibre and base. However, we shall give some easy (and well-known) examples that this condition is, in general, not sufficient for the existence of a symplectic structure on the total space, even if the structure group of the bundle preserves the symplectic form on the fibre. Furthermore, the known theorems on the existence of symplectic structures on fibre bundles (in particular Thurston's construction [9], which we shall use below) do not, in general, give any information about the cohomology classes that can be realized. Indeed, if we require in addition that the symplectic form ω be compatible with the fibration, i.e., that ω restricts to a symplectic form on each fibre F, then the cohomology classes a realized by symplectic forms on S^2-bundles over surfaces have to satisfy the restriction $a^2(M) > (a(F))^2$ if the bundle is nontrivial (see [6]).

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