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DIRECTIONAL DIFFERENTIABILITY OF THE ROTATION NUMBER FOR THE ALMOST-PERIODIC SCHRÖDINGER EQUATION

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1. Introduction. Let us consider the one-dimensional Schrödinger equation

$$L_E(x) = -x'' + g_0(t)x - Ex = 0 \tag{1}$$

with almost-periodic potential g_0. The set A of energies where the Lyapunov exponent vanishes is also known to be the essential support of the absolutely continuous part of the spectral measure. This paper deals with the variation of the rotation number on A, in particular, with the differentiability and Lipschitz character of this map.

Since $g_0(t)$ is a bounded function, then the boundaries $\pm\infty$ are of the limit-point type according to Weyl's classification. It follows that for E complex with $Im(E) > 0$, even if E belongs to the resolvent, L_E has a basis of solutions $u_+(t)$, $u_-(t)$ within $L^2(R_+)$ and $L^2(R_-)$. This allows us to define and represent the rotation number as an harmonic function on the upper half-plane. This is the starting point of a theory developed by Johnson, Moser, Kotani, Simon, and others. These authors obtain properties of the rotation number on the real axis taking limits following the lines $E + i\varepsilon$ as ε tends to 0^+. From this point of view the rotation number agrees with the imaginary part of the extension to the real axis of a Herglotz function. We give a new proof of some known facts and present several new results using a different technique which has some interest in itself.

Let us recall here some of the main notions and previous results needed to formulate our version. For the remainder of the paper, Ω will stand for the hull of g_0. The map $T: \Omega \times R \to R$; $(\xi, t) \to \xi_t$, where $\xi_t(s) = \xi(t + s)$, describes a continuous flow on Ω. Thus, (Ω, T) is a unique ergodic and almost-periodic flow. Actually, Ω is a compact group and the Haar measure its unique invariant measure. The map $J: \Omega \to R$ assigns its value at $t = 0$ to each function of Ω.

Let E be fixed. Pastur [24] has shown that all the equations

$$-x'' + \xi(t)x = Ex, \qquad \xi \in \Omega \tag{2}$$

admit a common description of the spectrum. Taking polar-symplectic coordinates $\varphi = \tan^{-1}(x/x')$, $r = 1/2\,(x^2 + (x')^2)$, we find the relations

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