Vol. 66, No. 2 DUKE MATHEMATICAL JOURNAL © May 1992

ON COEFFICIENT PROBLEMS FOR UNIVALENT FUNCTIONS AND CONFORMAL DIMENSION

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1. Introduction. Let **D** and **D*** denote respectively the open unit disk and $\{z: |z| > 1\}$. In this paper we introduce some new tools, one of which we call conformal dimension, to study coefficient problems for the classes

$$S_1 = \left\{ f = \sum_{n=1}^{\infty} a_n z^n : f \text{ is univalent on } \mathbf{D} \text{ and } \|f\|_\infty \leqslant 1 \right\},$$

$$S_2 = \left\{ f = z + \sum_{n=1}^{\infty} b_n z^{-n} : f \text{ is univalent on } \mathbf{D}^* \right\}.$$

For the class $S = \{f = z + \sum_{n=2}^{\infty} a_n z^n : f \text{ is univalent on } \mathbf{D}\}$, de Branges [2] has proved the Bieberbach conjecture, $|a_n| \leqslant n$. For S_1 and S_2 the exact decay rate is unknown. (In the classical literature, S_2 is called Σ.) For $f \in S_1$ the area of $f(\mathbf{D})$ is bounded by π, and consequently, $|a_n| \leqslant n^{-1/2}$. For $f \in S_2$ the estimate $|b_n| \leqslant n^{-1/2}$ is also true; this is known as the area theorem. (See Duren [8].) One might guess that these estimates mean the coefficient problems for S_1 and S_2 are closely related, and we shall see that this is so.

If $f \in S_1$, we can obtain an upper estimate for the size of a_n by writing the Cauchy estimate

$$(1.1) \qquad |a_n| \leqslant \frac{1}{n}\left(1 - \frac{1}{n}\right)^{-n+1} \int_0^{2\pi} \left| f'\left(\left(1 - \frac{1}{n}\right) e^{i\theta}\right) \right| \frac{d\theta}{2\pi}$$

$$\sim \frac{e}{n} \int_0^{2\pi} \left| f'\left(\left(1 - \frac{1}{n}\right) e^{i\theta}\right) \right| \frac{d\theta}{2\pi}$$

$$= \frac{e}{n} I\left(f', 1 - \frac{1}{n}\right).$$

The same estimate is valid for b_n when $f \in S_2$ if we simply replace the integral mean above by $I(f', 1 + 1/n)$. This philosophy was first used by Littlewood to show

Received 23 April 1990. Revision received 23 October 1991.
Jones supported by N.S.F. Grant DMS-86-02500.