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ABELIAN VARIETIES HAVING A REDUCTION OF $K3$ TYPE

YURI G. ZARHIN

§0. Abelian varieties of $K3$ type. Recall [22], [23] that an Abelian variety A over a finite field is of $K3$ type if either it is an ordinary elliptic curve or it has the same Newton polygon as the product of an ordinary elliptic curve and a $(\dim(A) - 1)$-dimensional supersingular Abelian variety. This means that its set of slopes is either $\{0, 1\}$ or $\{0, 1/2, 1\}$ and the slopes 0 and 1 have length 1. A special case of a theorem of Lenstra and Oort [9] asserts that, for each positive integer g and for each prime number p, there exists an absolutely simple g-dimensional Abelian variety of $K3$ type defined over a certain finite field of characteristic p. The property of $K3$ type is invariant under isogenies and extensions of base field.

Let A be a simple Abelian variety of $K3$ type. Then it is absolutely simple ([22, 2.7.0.]), and its endomorphism ring End A is commutative; i.e., $\mathrm{End}\, A \otimes \mathbb{Q}$ is a commutative field ([22, 2.7.1]). (In fact, it is a number field of degree $2\dim(A)$.) Since each Abelian variety can be lifted to characteristic zero [13], there exists a $\dim(A)$-dimensional Abelian variety Y defined over a certain number field such that its reduction at some place is good and isomorphic to A. Then there is a natural embedding of the endomorphism rings of Abelian varieties $\mathrm{End}\, Y \to \mathrm{End}\, A$. It follows easily that $\mathrm{End}\, Y \otimes \mathbb{Q}$ is also a commutative field; in particular, Y is absolutely simple. It is also possible to find a $\dim(A)$-dimensional absolutely simple Abelian variety Y' of CM-type defined over a certain number field such that its reduction at some place is good and *isogenous* to A [20]; in particular, this reduction is also an absolutely simple Abelian variety of $K3$ type. It follows from results of Oort [12] that there exists a 3-dimensional absolutely simple Abelian variety of $K3$ type which could not be lifted to an Abelian variety of CM-type in characteristic zero. (This gives a negative answer to a question of M. Borovoi.)

1. The main problem. Let X be an Abelian variety defined over a number field K. In this paper we study ℓ-adic Lie algebras $\mathfrak{g}_\ell = \mathfrak{g}_{\ell,X}$ attached to X in the following way [15], [10]. Let us fix an algebraic closure $K(a)$ of K and let $G(K) := \mathrm{Gal}(K(a)/K)$ be the Galois group of K. If m is a positive integer, their we write X_m for the group of elements $a \in X(K(a))$ such that $ma = 0$. It is known that X_m is a free $\mathbb{Z}/m\mathbb{Z}$-module of rank $2\dim(X)$. Let ℓ be a prime number. The Tate $\mathbb{Z}_\ell$-module $T_\ell(X)$ is defined as the projective limit of groups X_m, where m runs through all powers of ℓ and the

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