Vol. 65, No. 2 DUKE MATHEMATICAL JOURNAL © February 1992

REAL AND COMPLEX INTERPOLATION AND EXTRAPOLATION OF COMPACT OPERATORS

MICHAEL CWIKEL

In loving memory of Bernard John Cwikel, 1951–1991

0. Introduction. In 1960, Krasnosel'skiĭ [K] proved the following variant of the Riesz-Thorin theorem for linear operators T which have compactness properties: *If* $T: L^{p_0} \to L^{q_0}$ *boundedly and* $T: L^{p_1} \to L^{q_1}$ *compactly, where all four exponents are in the range* $[1, \infty]$ *and* $q_0 < \infty$, *then* $T: L^{p_\theta} \to L^{q_\theta}$ *compactly, where, as usual,* θ *is any number in* $(0, 1)$, $1/p_\theta = (1 - \theta)/p_0 + \theta/p_1$, *and* $1/q_\theta = (1 - \theta)/q_0 + \theta/q_1$. Soon after Krasnosel'skiĭ's work, there were major advances in transforming the ideas of the Riesz-Thorin theorem and also the Marcinkiewicz interpolation theorem into theories of abstract interpolation, applicable to the study of operators acting on arbitrary (Banach) spaces. In particular, the very important papers of Calderón [Ca] and Lions-Peetre [LP] appeared in 1964. The authors of both of these papers incorporated some Krasnosel'skiĭ-type results about compact operators into their theories: Suppose that $\bar{A} = (A_0, A_1)$ and $\bar{B} = (B_0, B_1)$ are two arbitrary compatible couples of Banach spaces and let T be a linear operator which maps A_0 into B_0 compactly and A_1 into B_1 boundedly. In the light of Krasnosel'skiĭ's theorem it is natural to ask two questions.

(R) Does it follows that T is also a compact map between the real-method interpolation spaces generated from (A_0, A_1) and (B_0, B_1) by the construction of [LP]?

(C) The same question for the complex-method interpolation spaces generated by the construction of [Ca].

Indeed, it was shown in [LP] that $T: (A_0, A_1)_{\theta,p} \to (B_0, B_1)_{\theta,p}$ is compact for each $\theta \in (0, 1)$ and $p \in [1, \infty]$ in the special cases when either $A_0 = A_1$ or $B_0 = B_1$. It was also shown in [Ca] that $T: [A_0, A_1]_\theta \to [B_0, B_1]_\theta$ is compact for each $\theta \in (0, 1)$ under special conditions on B_0 and B_1; namely, there must exist a net $\{\pi_\lambda\}_{\lambda \in \Lambda}$ of operators π_λ on $B_0 + B_1$, each mapping B_j boundedly into itself for $j = 0, 1$, and such that $\pi_\lambda(B_0)$ is finite-dimensional and $\lim_{\lambda \in \Lambda} \|\pi_\lambda b - b\|_{B_0} = 0$ for each $b \in B_0$. (See [Ca, Sections 9.6, 10.4] for a somewhat more general result of this type.)

A number of further papers dealing with the compactness of $T: (A_0, A_1)_{\theta,p} \to (B_0, B_1)_{\theta,p}$ or of $T: [A_0, A_1]_\theta \to [B_0, B_1]_\theta$ have appeared since 1964. The authors include Cobos, Edmunds, Fernandez, Kreĭn, Hayakawa, Peetre, Persson, Petunin, and Potter. But none of these results completely answer questions (R) or (C) since they each require some additional condition on the operator or the couples $\bar{A}$ or $\bar{B}$. For a survey of these papers and for references we refer to an interesting recent

Received 24 November 1990. Revision received 26 July 1991.
Research supported by the Technion V.P.R. Fund—Argentinian Research Fund.