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CONTRACTION OF NONSINGULAR CURVES

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1. Introduction. Let Y be a smooth Riemann surface of genus g lying inside a (not necessarily smooth) analytic space X. In this paper we give a cohomological criterion for contracting Y to a point, that is, for constructing an analytic morphism

$$f: X \to X'$$

such that $f(Y) = \{x'\}$ and $f|_{X \setminus Y}$ is an isomorphism onto $X \setminus \{x'\}$. Let $(\hat{X}, \mathcal{O}_{\hat{X}})$ be the formal neighborhood of X along Y. For any coherent sheaf $\mathscr{F}$ on X, let $\hat{\mathscr{F}}$ be the completion of $\mathscr{F}$ along Y and $Gr_Y^0(\mathscr{F}) := (\mathscr{F}/\mathscr{I}_Y\mathscr{F})/(\text{torsion})$. Here, $\mathscr{I}_Y$ is the ideal sheaf defining the reduced structure on Y.

Notice that, because Y is one-dimensional and nonsingular, $Gr_Y^0(\mathscr{F})$ is locally free. This fact is a very important ingredient in the proof of the next theorem. If Y has dimension greater than one, this is no longer true in general; so it would be interesting to find another way to deal with that case.

THEOREM 1. *$Y \subseteq X$ contracts to a point if and only if there is a line bundle $\mathscr{L}$ on a neighborhood of Y in X such that*

$$\deg(\mathscr{L}|_Y) \leqslant g - 2$$

and $H^1(\hat{X}, \hat{\mathscr{L}})$ is finite-dimensional.

In Section 4 we will show that this theorem gives restrictions on the second-order neighborhood of Y in X. We will also show that, if Y is rational and the degree of the normal bundle of Y in X is sufficiently negative, there are examples where Y contracts to a point. We will also see that, when X is a smooth 3-fold along Y, the construction gives most of the examples of contractible curves. See Section 4 for more detail.

Before stating the second theorem, we need to give some definitions. An *adic ring* R is a Noetherian ring R endowed with a topology defined by an ideal I such that this topology is separated and complete. A homomorphism of adic rings is said to be an *adic morphism* if it is continuous. We say that Y has a *rational formal neighborhood in* X if $H^1(\hat{X}, \mathcal{O}_{\hat{X}}) = 0$. Thus, if Y has a rational formal neighborhood in X, we must have $H^1(Y, \mathcal{O}_Y) = 0$ so that Y has to be rational. Also, in this situation $H^1(Y, Gr_Y^0(\mathscr{I}_Y)) = 0$ so that $Gr_Y^0(\mathscr{I}_Y) = \bigoplus_{0 \leqslant 1 \leqslant \dim X - 1} \mathcal{O}_Y(a_i)$, where all $a_i \geqslant -1$. Next, we define a sequence of ideal sheaves as follows.

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