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ISOSPECTRALITY AND COMMENSURABILITY OF ARITHMETIC HYPERBOLIC 2- AND 3-MANIFOLDS

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0. Introduction. Let M be a compact Riemannian manifold. The eigenvalues of the Laplace operator Δ on the space of $L^2(M)$ functions form a discrete sequence in $\mathbb{R}$: $0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots$, and the collection of values $\{\lambda_i\}$ together with the multiplicities with which they occur is called the *spectrum of the Laplacian* of M. Two compact Riemannian manifolds are called *isospectral* if the spectra of their Laplacians are identical.

The question as to whether there exist isospectral but nonisometric manifolds has received considerable attention; the first examples of such manifolds were two flat 16-dimensional tori constructed by Milnor [Mi] in 1964, and in the subsequent years many more classes of examples were constructed. In particular, Vignéras in [V1] constructed the first examples of hyperbolic 2- and 3-manifolds (by which we shall always mean hyperbolic 2- and 3-manifolds of constant curvature -1). Both the methods of Milnor and Vignéras relied on reformulating the problem of finding isospectral but nonisometric manifolds in a number theoretic setting. A more general method of producing examples was given by Sunada in [S], and this method has been used to great effect to produce large numbers of examples. See, for example [Br], [Bu], and [Sp].

To date, these two methods are the only known methods of constructing isospectral but nonisometric manifolds. Indeed, it is known that these methods produce distinct classes of isospectral manifolds (when the methods of [V1] apply); see [C]. Both these constructions produce manifolds which are commensurable. It is an intriguing question as to whether isospectrality implies commensurability. The main result of this paper is Theorem 2.1 which shows that this is indeed the case for *arithmetic* hyperbolic 2- and 3-manifolds (see §2 for a definition), thereby providing the first real evidence of a positive answer to the above question.

The remainder of the paper is arranged as follows. In Section 1 we collect some facts concerning the length spectrum (resp. complex length spectrum) of a hyperbolic manifold (resp. hyperbolic 3-manifold). In Section 2 we prove the main result. In Section 3 we take up a theme developed by Spatzier in [Sp] where he showed that isospectral but nonisometric hyperbolic manifolds were to be found in great abundance in large enough dimensions. In Section 3 we prove that this is also the case in dimension 3; our arguments are distinct from those of [Sp], but Sunada's method is at the core of the construction once again. The methods used are of independent interest as they develop techniques for computing the commensurability subgroup

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