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SURFACES OF DEGREE d WITH SECTIONAL GENUS g IN $\mathbb{P}^{d+1-g}$ AND DEFORMATIONS OF CONES

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0. Introduction. In this paper we first review the relation between the theory of normal surfaces $S_d \subset \mathbb{P}^{d+1-g}$ of degree d with a hyperplane section of genus g and the theory of the deformations of cones over curves. Our main result is the complete classification of these surfaces S_d with $d \geqslant 4g - 4$. Along the way, we answer some questions about the deformations of cones over curves. In particular, we extend Pinkham's work [Pi1] on deformations of cones over curves of genus 0 and 1, to nonhyperelliptic curves of genus $\geqslant 3$. The deformations of cones over hyperelliptic curves have been studied in [T3].

Let X be a smooth complex projective curve of genus g and $X \subset \mathbb{P}^N = \mathbb{P}(H^0(M))$ be the embedding by a very ample line bundle M of degree d. Consider the natural map $\theta: \mathbb{C}^{N+1} - \{0\} \to \mathbb{P}^N$. Define the affine cone of X to be $\theta^{-1}(X) \cup \{0\}$ and the projective cone of X to be the projective closure of the affine cone of X in $\mathbb{P}^{N+1}$. When there is no ambiguity in context, we will denote both the affine and projective cones of X by (X, M). A useful tool for studying the deformations of the affine cone of X when $d \geqslant 4g - 4$ and $M \neq 2K_X$ is the correspondence developed in [Pi1, Thms. 4.2 and 5.1] between the deformations of the affine cone of X and the deformations of the projective cone of X in $\mathbb{P}^{d+1-g}$. From this point of view it is natural to look at all normal surfaces $S_d \subset \mathbb{P}^{d+1-g}$ of degree d with a hyperplane section of genus g. For example, suppose that X is a hyperplane section of the normal surface $S_d \subset \mathbb{P}^{d+1-g}$ and M is the normal bundle of X in S_d. If S_d is arithmetically Cohen-Macaulay, then by [Pi2, Thm. 6.7] the projective and affine cones (X, M) can be deformed to S_d and $S_d - X$, respectively, by sweeping out the cone over S_d with hyperplane sections passing through X. We call this *Pinkham's construction.* Note that one can deform the cone (X, M) projectively either by keeping X and M fixed in the deformations (e.g., in Pinkham's construction) or by varying X or M. These are known as deforming the cone *strictly negatively* or *seminegatively*, respectively. We will give examples of singularities which smooth seminegatively but not strictly negatively. (See 4.10 and 5.6.)

Let $\varphi: V \to S_d \subset \mathbb{P}^{d+1-g}$ be the minimal resolution of the normal surface S_d of degree d with a hyperplane section of genus g. We find that, if $d \geqslant 2g + 1$, then φ is the morphism associated to the complete linear system $|L|$, where $L := \varphi^*(\mathcal{O}_{S_d}(1))$ and V is a ruled surface. Furthermore, S_d cannot be a smooth irregular surface. Using a result of Horowitz [Ho, Cor. 1.8], we then deduce the following theorem.

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