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THE THOM CONDITION ALONG A LINE

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§0. Introduction. Let $f: (\mathbb{C} \times \mathbb{C}^{n+1}, \mathbb{C} \times \mathbf{0}) \to (\mathbb{C}, 0)$ be a polynomial and let Σf denote the set of critical points of the map f. Let $\mathbf{p}_i$ be a sequence of points in $\mathbb{C}^{n+2} - \Sigma f$ such that $\mathbf{p}_i \to \mathbf{0}$ and $T = \lim T_{\mathbf{p}_i} V(f - f(\mathbf{p}_i))$ exists. Then, $\mathbb{C} \times \mathbf{0}$ is said to satisfy the Thom condition [3] (or the a_f condition [13]) at the origin if T necessarily contains $T_\mathbf{0}(\mathbb{C} \times \mathbf{0})$.

If f defines a family of isolated singularities, $f_t: (\mathbb{C}^{n+1}, \mathbf{0}) \to (\mathbb{C}, 0)$ then, in [9], Lê and Saito give a numerical criterion which guarantees that $\mathbb{C} \times \mathbf{0}$ satisfies the Thom condition at the origin: if the Milnor number of f_t is constant for all t small, then $\mathbb{C} \times \mathbf{0}$ satisfies the Thom condition at the origin. In [10], we proved the analogous result for families of one-dimensional singularities. Namely, there are two numbers —which we now denote by $\lambda^0_{f_t}$ and $\lambda^1_{f_t}$—whose constancy for all t small implies that $\mathbb{C} \times \mathbf{0}$ satisfies the Thom condition at the origin.

In this paper, we generalize this result to families of singularities of arbitrary dimension. More precisely, if $s = \dim_\mathbf{0} \Sigma V(f_0)$, then we define a collection of numbers (the Lê numbers [11]), $\lambda^0_{f_t}, \ldots, \lambda^s_{f_t}$, whose constancy for all t small implies that $\mathbb{C} \times \mathbf{0}$ satisfies the Thom condition at the origin. *It is important to note that we do this without any further assumptions on how generic the coordinate t must be*—that is, the existence and constancy of the Lê numbers implies that the coordinate t (actually, the hyperplane $V(t)$) is sufficiently generic to reach the desired conclusion. This is crucial if one wishes to study deformations of some particular f_0.

§1. The Thom Set. We continue with $f: (\mathbb{C} \times \mathbb{C}^{n+1}, \mathbb{C} \times \mathbf{0}) \to (\mathbb{C}, 0)$ a polynomial.

Definition 1.1. The **Thom set of f at the origin,** $\mathscr{T}_f$, is the set of $(n+1)$-planes which occur as limits at the origin of tangent planes to level hypersurfaces of f, i.e., $T \in \mathscr{T}_f$ if and only if there exists a sequence of points $\mathbf{p}_i$ in $\mathbb{C}^{n+2} - \Sigma f$ such that $\mathbf{p}_i \to \mathbf{0}$ and $T = \lim T_{\mathbf{p}_i} V(f - f(\mathbf{p}_i))$. Equivalently, $\mathscr{T}_f$ is the fibre over the origin in the Jacobian blow-up of f (see [5]). $\mathscr{T}_f$ is thus a closed algebraic subset of the Grassmanian $G_{n+1}(\mathbb{C}^{n+2})$ = the projective space of $(n+1)$-planes in $\mathbb{C}^{n+2}$.

We define $\mathscr{T}_f^{\text{anal}}$ to be the set of $(n+1)$-planes which occur as limits at the origin of tangent planes to level hypersurfaces of f *as we approach the origin along real analytic paths*, i.e., $T \in \mathscr{T}_f^{\text{anal}}$ if and only if there exists a real analytic path $\alpha: [0, \varepsilon) \to \{\mathbf{0}\} \cup (\mathbb{C}^{n-2} - \Sigma f)$ such that: $\alpha(u) = \mathbf{0}$ if and only if $u = 0$, and $T = \lim_{u \to 0} T_{\alpha(u)} \cdot V(f - f(\alpha(u)))$. Clearly, $\mathscr{T}_f^{\text{anal}} \subseteq \mathscr{T}_f$. In fact,

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