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ON THE CONSTRUCTION OF SOME COMPLETE METRICS WITH EXCEPTIONAL HOLONOMY

ROBERT L. BRYANT AND SIMON M. SALAMON

1. Introduction. The first author established in [Br] the existence of Riemannian metrics on open sets of R^7 and R^8 with holonomy group equal to G_2 and Spin(7), respectively. These two groups constituted the two exceptional members of Berger's list of holonomy groups of irreducible Riemannian manifolds whose existence had remained in doubt [A], [B], [M]. In common with the other groups SU(n) and Sp(n) in the list that do not also arise as holonomy groups of symmetric spaces, G_2 and Spin(7) have the property that their metrics are automatically Ricci-flat [Bo]. For background on holonomy groups we also refer the reader to [Be].

Although the existence question was first settled by analysis of a suitable differential system, [Br] also included an example of a metric with holonomy G_2 on $R^+ \times M^6$, and one with holonomy Spin(7) on $R^+ \times M^7$, where M^6 and M^7 are certain homogeneous spaces of the indicated dimension. It was partly a deeper understanding of the first of these examples which led to the present paper, in which we explicitly construct essentially three distinct complete metrics with holonomy equal to G_2, one complete metric with holonomy equal to Spin(7), and various other incomplete metrics with exceptional holonomy.

The metrics are all encountered on total spaces of vector bundles over manifolds of dimension 3 and 4. For G_2, the basic idea is to consider 7-manifolds with an SO(3)- or SO(4)-structure corresponding to inclusions $SO(3) \subset SO(4) \subset G_2$, and a splitting of dimensions $7 = 3 + 4$. On these manifolds one seeks a 3-form ψ satisfying $d\psi = 0 = d * \psi$, a condition which, as we explain in section 2, characterizes the holonomy reduction. Accordingly, we study in section 3 the spin bundle S (fiber R^4) over a 3-dimensional space form M^3 with either positive or negative constant sectional curvature. A description (using quaternions) of invariant forms on the total space of S enables us to single out a family of metrics whose holonomy actually equals G_2. We go to some trouble to prove the equality, which amounts to checking irreducibility. One of these metrics is complete for $M^3 = S^3$.

In Section 4, similar techniques are applied to the bundle Λ^2_- of anti-self-dual 2-forms (fiber R^3) over a self-dual Einstein 4-manifold M^4. The hypotheses on M^4 ensure that the curvature of Λ^2_- is entirely determined by the scalar curvature of M^4, which we take to be either positive or negative. Sympathetic differential relations between invariant forms then lead to metrics of the desired type on Λ^2_-.

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