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WITT GROUPS OF AFFINE THREE-FOLDS

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Let R be a regular affine algebra over the field of real numbers. Knebusch in [12] raised the question whether the Witt group $W(R)$ of R is finitely generated. This question was answered in the affirmative for algebras of dimension one by Knebusch himself ([13]) and for algebras of dimension two by M. Ayoub ([3]). The aim of this paper is to show that for regular affine domains R of dimension three over the field of real or complex numbers, finite generation of $W(R)$ is equivalent to the finiteness of $CH^2(R) \otimes \mathbb{Z}/2\mathbb{Z}$, where $CH^2(R)$ denotes the group of codimension-two algebraic cycles of $\operatorname{spec} R$ modulo rational equivalence. Over $\mathbb{C}$, finiteness of $CH^2(R) \otimes \mathbb{Z}/2\mathbb{Z}$, though expected to be true, is still an open question and our result, which links this question to one on quadratic forms, seems to be of independent interst. For unirational affine 3-folds $\operatorname{spec} R$, $CH^2(R) \otimes \mathbb{Z}/2\mathbb{Z}$ is finite, so that $W(R)$ is finitely generated.

Here is a brief description of our method of proof. Let X be a smooth integral scheme with 2 invertible in $\Gamma(X, \mathcal{O}_X)$. Let K denote the function field of X. Let $\{I_n(X)\}_{n\geqslant 0}$ denote the filtration on $W(X)$ which is the pull-back of the filtration $\{I^n(K)\}_{n\geqslant 0}$ on $W(K)$ under the map i: $W(X) \to W(K)$ induced by the generalization $\operatorname{spec} K \to X$. Let $\Gamma(X, \mathcal{H}^n)$ denote the group of Zariski sections of the sheaf associated to the presheaf $U \mapsto H^n_{\text{et}}(U, \mu_2)$. We construct homomorphisms of $I_n(X)$ into $\Gamma(X, \mathcal{H}^n)$, for a certain class of schemes which includes all smooth algebraic schemes of dimension $\leqslant 3$ over $\mathbb{R}$ or $\mathbb{C}$, using the results of [2] and [5]. (Indeed, if the invariant problem 1.4 in [2] has an affirmative answer, these homomorphisms are defined for all integral schemes.) We use the results of [5] to study the finiteness of the groups $\Gamma(X, \mathcal{H}^n)$ if $\dim X \leqslant 3$. For real 3-folds, the equivalence of the finiteness of $CH^2(X) \otimes \mathbb{Z}/2$ and the finite generation of $W(X)$ is a consequence of the finiteness of the group $\Gamma(X, \mathcal{H}^4)$. This is proved in [6] by constructing a signature homomorphism $\Gamma(X, \mathcal{H}^4) \to C(\operatorname{spec}_r X, \mathbb{Z}/2\mathbb{Z}) \simeq (\mathbb{Z}/2\mathbb{Z})^s$ and showing that it is injective. (Here, s denotes the number of connected components of $X(\mathbb{R})$ for the Euclidean topology and $\operatorname{spec}_r X$ denotes the real spectrum of X ([14]).)

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