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THE POISSON BRACKET ON THE SPACE OF MEASURED FOLIATIONS ON A SURFACE

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W. Thurston showed that if M is a closed oriented and connected surface, it is possible to define a piecewise symplectic structure on $\mathcal{MF}(M)$ the space of classes of measured foliations up to Whitehead moves and isotopy—see the work of A. Papadopoulos [Pa1] and [Pa2]. There are natural functions defined on $\mathcal{MF}(M)$, if θ is an unoriented homotopy class of closed curves and $\mathcal{F} \in \mathcal{MF}(M)$ the number $i(\theta, \mathcal{F})$ is defined as the minimum of the total transverse measure of a curve representing θ with respect to a foliation representing $\mathcal{F}$. In the case $\theta \in \mathcal{S}(M)$, the set of isotopy classes of simple closed curves, the properties of this intersection number are in [FLP]. It is possible to generalize the properties to a nonsimple θ; for example, it is possible to show that the function $\theta \mapsto i(\theta, \mathcal{F})$ is continuous—see [Bo] for proofs in terms of geodesic laminations. We denote by $\tilde{\theta}$ the function $\mathcal{MF}(M) \to \mathbb{R}_+$, $\mathcal{F} \mapsto i(\theta, \mathcal{F})$.

In [Pa1] it is shown that if $\gamma \in \mathcal{S}(M)$, there exists a Hamiltonian flow $\lambda \in \mathbb{R} \mapsto H_\gamma^\lambda$ that is defined on the open set $\{\mathcal{F} | i(\gamma, \mathcal{F}) \neq 0\}$, whose Hamiltonian is precisely $\tilde{\gamma}$. It is easy to remark that $H_\gamma^{i(\gamma, \mathcal{F})}(\mathcal{F})$ is the image of $\mathcal{F}$ under the positive Dehn twist defined by γ.

Our goal is to compute the Poisson bracket $\{\tilde{\gamma}, \tilde{\delta}\}$ for $\gamma, \delta \in \mathcal{S}(M)$. Recall that $\{\tilde{\gamma}, \tilde{\delta}\}$ is a function which is defined as the derivative at 0 of $\lambda \mapsto -\tilde{\delta}(H_\gamma^\lambda(\mathcal{F})) = -i(\delta, H_\gamma^\lambda(\mathcal{F}))$.

Following W. Goldman [Go], for each pair $\gamma, \delta \in \mathcal{S}(M)$ we introduce $i(\gamma, \delta)$ pairs of homotopy classes of unoriented closed curves on M, namely $\{(\theta_{j\gamma\delta}^+, \theta_{j\gamma\delta}^-) | j = 1, \dots, i(\gamma, \delta)\}$. To define these curves, choose representatives $c \in \gamma$ and $d \in \delta$ such that $c \cap d$ has exactly $i(\gamma, \delta)$ points and the curves are transverse at all points of intersection. Call $p_1, \dots, p_{i(\gamma, \delta)}$ the points in $c \cap d$. For each $j = 1, \dots, i(\gamma, \delta)$ we define two curves $t_{j\gamma\delta}^+$ and $t_{j\gamma\delta}^-$ in the following way: we choose an orientation on c and an orientation on d such that the tangent vector of c at p_j followed by the tangent vector of d at p_j gives the orientation of M at p_j; consider c (resp. d) as a curve c_{p_j} (resp. d_{p_j}) pointed at p_j; and define $t_{j\gamma\delta}^+$ (resp. $t_{j\gamma\delta}^-$) as the curve $c_{p_j} d_{p_j}$ (resp. $c_{p_j} d_{p_j}^{-1}$). Remark that if we change the orientation on c, we must also change it on d, and this replaces $c_{p_j} d_{p_j}$ by $c_{p_j}^{-1} d_{p_j}^{-1} = d_{p_j}(c_{p_j} d_{p_j})^{-1} d_{p_j}^{-1}$, which is freely homotopic to $t_{j\gamma\delta}^+$ as unoriented curve. If we take other representatives $c' \in \gamma$ and $d' \in \delta$ such that $c' \cap d'$ consists of $i(\gamma, \delta)$ points, then, by [FLP, Exposé 3, Proposition 12, page 48], there exists a diffeomorphism isotopic to the identity which sends c to c' and d to d'.

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