Vol. 55, No. 2 DUKE MATHEMATICAL JOURNAL © June 1987

GEODESICS IN HOMOLOGY CLASSES

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§1. Introduction. Let M be a compact Riemannian manifold of negative curvature and let $\pi(x)$ denote the number of prime closed geodesics on M whose length is at most x. A well known result due to Margulis [M] asserts that asymptotically

$$\pi(x) \sim ce^{hx}/x; \tag{1.1}$$

here h is the topological entropy of the geodesic flow and c is a suitable positive constant. We call such a result a prime geodesic theorem. Recently Adachi and Sunada [A] studied the following interesting prime geodesic problem: If $\phi\colon \Gamma \to H_1(M, \mathbb{Z})$ is the projection of the fundamental group onto the first homology group, then for $\beta \in H_1$ let $\pi_\beta(x)$ be the number of primitive closed geodesics γ on M of length at most x for which $\phi(\gamma) = \beta$. They prove that

$$\lim_{x \to \infty} \frac{\log \pi_\beta(x)}{x} = h. \tag{1.2}$$

Our aim in this note is to prove a more detailed prime geodesic theorem for $\pi_\beta(x)$ when M is an n-dimensional hyperbolic manifold. Note that $\pi_\beta(x)$ counts the number of prime geodesics of length at most x whose algebraic winding about a generating set of cycles is exactly β. As pointed out by Adachi and Sunada the most interesting case is when $H_1(M, \mathbb{Z})$ is infinite for then the nature of the asymptotics should change. Our analysis shows that this is so and, in fact, that the rank of $H_1(M, \mathbb{Z})$ characterizes the asymptotic behaviour.

Let $\psi\colon \Gamma \to \Lambda$ be a surjective homomorphism of the fundamental group Γ of M onto an abelian group Λ. Let r be the rank of Λ and let m be the order of the torsion of Λ; i.e., Λ is isomorphic to $\mathbb{Z}^r \cdot F$, F being finite of order m.

THEOREM. *For $\beta \in \Lambda$ we have the asymptotic expansion*

$$\pi_\beta(x) \sim \frac{e^{(n-1)x}}{mx^{r/2+1}}\left(c_0 + c_1/x + c_2/x^2 \dots\right) \tag{1.3}$$

as $x \to \infty$. Here $c_0 > 0$ and is independent of β; c_0 is the determinant of a certain period matrix of harmonic 1-forms on M.

Received September 8, 1986. The work of the first author was supported in part by NSF under Grant No. DMS-85-03297 and the second by NSF Grant No. DMS-85-04329.