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C^m APPROXIMATION BY SOLUTIONS OF ELLIPTIC EQUATIONS, AND CALDERÓN-ZYGMUND OPERATORS

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0. Introduction. A sample of the results to be found in this paper is the following

THEOREM. *Let $X \subset \mathbb{R}^n$ be compact and let f be a twice continuously differentiable function on a neighbourhood of X, whose Laplacian vanishes at each point of X. Then there is a sequence (f_j) such that each f_j is a harmonic function on a neighbourhood (depending on j) of X, and*

$$\partial^\alpha f_j \to \partial^\alpha f \quad \text{uniformly on } X, \text{ for } 0 \leqslant |\alpha| \leqslant 2.$$

A remarkable feature of the above theorem is that it depends on an apparently unrelated and deep result in Fourier analysis: the weak L^1 type estimate for (homogeneous) Calderón-Zygmund operators. Such an estimate is used indirectly via a theorem of Nguyen [8] on the action of Calderón-Zygmund operators on bounded functions. The other main ingredient in the proof is Vitushkin's localization and coefficient matching technique for planar holomorphic approximation [16], such as adapted to elliptic approximation problems by Bagby [1] and O'Farrell [11]. In fact our methods work for constant coefficient homogeneous elliptic operators and give the best possible results for other smoothness degrees of approximation. In order to describe the setting for our theorems we now introduce some notation.

For a multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$ with $0 \leqslant \alpha_j \in \mathbb{Z}$ we let $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\alpha! = \alpha_1! \dots \alpha_n!$, $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$ for $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $\partial^\alpha = (\partial/\partial x_1)^{\alpha_1} \dots (\partial/\partial x_n)^{\alpha_n}$. Throughout the paper we will consider a fixed homogeneous polynomial of degree r, with complex coefficients,

$$L(\xi) = \sum_{|\alpha|=r} a_\alpha \xi^\alpha, \qquad \xi \in \mathbb{R}^n,$$

which satisfies the ellipticity condition

$$L(\xi) \neq 0, \qquad \xi \neq 0,$$

and we will associate to it the homogeneous elliptic operator

$$L = L(\partial) = \sum_{|\alpha|=r} a_\alpha \partial^\alpha.$$

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