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SOME ASPHERICAL MANIFOLDS

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0. Introduction. Let Y denote the vector space of real, tridiagonal,[1] symmetric, $(n+1) \times (n+1)$ matrices. Let Λ be any set of $n+1$ distinct real numbers. Let P^n denote the set of those matrices in Y with spectrum equal to Λ. In [T], Carlos Tomei proves that the space P^n is a closed n-manifold. Of course, this is hardly surprising; however, Tomei goes on to show that these manifolds have several amazing properties. The most surprising property is that P^n is aspherical: in fact, its universal cover is diffeomorphic to Euclidean n-space. P^1 is a circle. P^2 is a surface of genus two. P^3 is the "double" of a certain hyperbolic 3-manifold of finite volume.[2] The proof of the asphericity of P^n in [T] uses results from [D1] on groups generated by reflections. Before continuing our description of these manifolds, we need to make a few general remarks concerning reflection groups.

Suppose that W is a discrete group acting smoothly and properly on a manifold M and that W is generated by smooth reflections. A *chamber* X for W on M is the closure of a component of the set of nonsingular points. Let S denote the set of reflections on W across the codimension-one faces of X. Then (W, S) is a Coxeter system (cf. [D1]). The manifold M can be reconstructed from the chamber X and the group W: paste together copies of X, one for each element of W, in the obvious fashion. In [D1], we gave simple necessary and sufficient conditions for the result of this pasting construction to be contractible.

There is a natural group generated by reflections on Tomei's manifold P^n. This can be seen as follows. The group $O(n+1)$ acts by conjugation on the vector space of $(n+1) \times (n+1)$ symmetric matrices. The kernel of this action is $\{\pm 1\}$. Let J denote the group {diagonal matrices in $O(n+1)\}/\{\pm 1\}$. Obviously, $J \cong (\mathbb{Z}/2)^n$. The subspace Y is J-stable. J acts on Y as the group of all possible sign changes of the off-diagonal entries. Thus, J is a linear reflection group on Y. Since $O(n+1)/\{\pm 1\}$ preserves the spectrum of a symmetric matrix, the submanifold P^n is J-stable and J is a smooth reflection group on it. A fundamental chamber X^n for J on P^n is the intersection of P^n with the set of

[1] To say that a matrix $y = (y_{ij})$ is "tridiagonal" means that $y_{ij} = 0$ whenever $|i - j| > 1$.

[2] This means that there is a compact 3-manifold M^3 such that (a) each component of ∂M^3 is torus. (b) the interior of M^3 is homeomorphic to a hyperbolic 3-manifold of finite volume, and (c) P^3 is the double of M^3.

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