Vol. 50, No. 3 DUKE MATHEMATICAL JOURNAL © September 1983

HYPOELLIPTICITY OF A SYSTEM OF COMPLEX VECTOR FIELDS

MEI-CHI SHAW

In this paper we study the hypoellipticity of a system of complex vector fields. Suppose we are given vector fields $L_1, \ldots, L_m$ in a neighborhood R^N where

$$L_i = \sum_{k=1}^{N} a_i^k D_k$$

where a_i^k are complex-valued C^∞ functions. We wish to study the hypoellipticity of the system

$$L_i u = f_i \qquad i = 1, \ldots, m \tag{0.1}$$

i.e., when is u smooth if every f_i is smooth. This question arises when we study the hypoellipticity of tangential Cauchy–Riemann equations for CR structure. When (0.1) has only trivial characteristic set (see Definition 1.1), the system is elliptic which obviously implies hypoellipticity. Thus we are interested in the case when the characteristic set has dimension greater or equal to one. When the L_i's in (0.1) are the tangential Cauchy–Riemann equations for some hypersurface in $\mathbf{C}^n$, which implies that the dimension of the characteristic set is equal to one, Kohn [10] has given a sufficient condition for hypoellipticity of such a system. His work is a special case of our result, since we do not impose Frobenius conditions on the system (0.1) or restrict the dimension of the characteristic set.

After completing the proofs in this paper, the author has been informed by A. Melin that if the problem is microlocalized, our result is contained in Egorov's theorem [4] (also see Hörmander [7] and Treves [11]) for the hypoellipticity of a single vector field. However, Egorov used the method of Fourier integral operator to transform the vector field into a simpler form. In contrast, our more direct approach avoids the tedious machinery of the Fourier integral operator. Thus it offers more insight into the problem and suggests possible application to related problems.

This work was inspired by the recent works of Baouendi, Chang and Treves [1] and of Boggess and Polking [3] on the holomorphic extension of CR functions. Of the three equivalent conditions for "condition Z" in Definition 1.3, it was proved in [1] that under the assumption of condition I and integrability, the system is analytic hypoelliptic, which implies the extendability of the CR

Received December 17, 1982. Revision received February 7, 1983.