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SPREADING OF SINGULARITIES FOR A SEMILINEAR WAVE EQUATION

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0. Introduction. For the linear wave equation

$$\Box u \equiv \left(\frac{\partial^2}{\partial t^2} - \sum_1^n \frac{\partial^2}{\partial x_i^2} \right) u = 0$$

the wave front set of u is invariant under the Hamiltonian flow (Hörmander [3]) so in particular

the singular support of u is contained in the union of the light cones over the singular support of the initial data. (0.1)

The purpose of this paper is the construction of a solution of the equation $\Box u = u^3$ with $u(-t_0) \in H^s(R^n)$, $u_t(-t_0) \in H^{s-1}(R^n)$, $u \in H^s((-t_0, t_0) \times R^n)$, such that (0.1) fails for $t > 0$ (Theorem 1). Here $n \geqslant 2$, $s > (n+1)/2$.

In one space dimension, (0.1) holds for the general semilinear equation $\Box u = f(u, u_x, u_t)$ (Reed [10]). If $n = 1$ and the order of the semilinear strictly hyperbolic equation is greater than two, (0.1) fails in general. When two characteristics carrying singularities cross, new singularities can propagate from the crossing point along all the other characteristics (Rauch–Reed [8], [9]). If $n = 1$ and the order is two there are no additional directions, but for $n > 1$ there are, and it has been conjectured for some time that the singularities do spread for second order equations. An example for $n \geqslant 3$, $\Box u = u^2$, was announced in Lascar [6].

Let T_1 be an operator satisfying $\Box(T_1 w) = w$, $t \geqslant -t_0$, $T_1 w \equiv 0$ for $t \ll 0$, and let $v \in H^s$ satisfy $\Box v = 0$. Then $u = v + T_1 u^3$ satisfies $\Box u = u^3$, $t \geqslant -t_0$. Writing $u = v + T_1 v^3 + T_1(u^3 - v^3)$, the idea is to construct v with $WF(v^3) \not\subset WF(v)$ at $t = 0$, such that

for $t > 0$, $T_1 v^3$ has singularities not present in v, microlocally in H^{3s-n+2} and not in $H^{3s-n+2+\epsilon}$, while (0.2)

if $T_1(u^3 - v^3)$ has new singularities, microlocally these are in $H^{3s-n+(5/2)}$. (0.3)

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