Vol. 48, No. 4 DUKE MATHEMATICAL JOURNAL © December 1981

THE PROOF OF THE CENTRAL LIMIT THEOREM FOR THETA SUMS

W. B. JURKAT AND J. W. VAN HORNE

§1. Introduction. For $x \in [0,1]$ and $N > 0$ real we consider the finite theta sums

$$S_N(x) = 1/2 + \sum_{0<n<N} \exp(i\pi n^2 x) + (1/2)\exp(i\pi N^2 x)$$

where we include the last term only if N is an integer. The pointwise behavior of these sums was investigated initially by Hardy and Littlewood [3] and most recently by Fiedler, Jurkat and Körner [2]. We are interested in the sequence of distribution functions $\Phi_N(\lambda) = |\{x \in [0,1] : \lambda \leqslant |S_N(x)|N^{-1/2}\}|$ for $\lambda \geqslant 0$ where $|\cdot|$ denotes Lebesgue measure. In a previous paper [4] we conjectured that Φ_N converges to a limiting distribution, but we could only prove partial results in that direction. They were based on an asymptotic expansion of S_N up to an error $O(\sqrt{N})$ given in [2]. Now we are able to prove the existence of a limiting distribution using an improved approximation formula (Theorem 2) with error $o(\sqrt{N})$ or even smaller if desired. The convergence is based upon abstract principles, described in Theorem 1, which, while explaining the existence of the limit, unfortunately cannot immediately be used to calculate it. In combination with Theorem 2 this yields the "central limit theorem" for theta sums (Theorem 3). Although it bears some resemblance to the central limit theorem for nearly independent random variables (cf. [5], [6], and [8]), it is considerably different, primarily due to the lack of independence, and therefore of considerable interest as a first example of this kind. An application of this result is the determination of the asymptotic behavior of the moments $\int_{-1}^{1} |S_N(x)|^\alpha\, dx$ as N tends to ∞ for all $\alpha > 0$ (Theorems 4 and 5). Similar results can also be obtained for the theta series $\theta(s) = \sum_{n=-\infty}^{\infty} \exp(-\pi n^2 s)$ ($s = \sigma + it$, $\sigma > 0$), i.e. for $\int_{-1}^{1} |\theta(\sigma + it)|^\alpha\, dt$ as $\sigma \to 0+$ (Theorem 6). These results are somewhat simpler, however.

Much of the notation is as in [4]. We let $E(x) = \exp(i\pi x)$. For functions f and g we write $f(x) = O(g(x))$ when there is a constant B such that $|f(x)| \leqslant B|g(x)|$ for all x in the given range. Unless otherwise indicated, B is an absolute constant. By $f(x) \sim g(x)$ as $x \to x_0$ we mean $\lim_{x \to x_0} f(x)/g(x) = 1$, where x_0 may be ∞. For any measurable function $h(x)$ defined on $[0,1]$ we let $D(\lambda; h) = |\{x \in [0,1] : \lambda \leqslant |h(x)|\}|$ denote its distribution function for $\lambda \geqslant 0$. If $c(q,r)$ is an array of complex numbers defined for certain integers q and r, and B is a Riemann

Received April 6, 1981. Revision received September 10, 1981.