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SINGULAR VARIATION OF DOMAINS AND EIGENVALUES OF THE LAPLACIAN

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§1. Introduction. Let Ω be a bounded domain in $\mathbf{R}^n$ with C^∞ boundary γ and w be a fixed point in Ω. For any sufficiently small $\epsilon > 0$, let B_ϵ be the ball defined by

$$B_\epsilon = \{z \in \Omega; \quad |z - w| < \epsilon\}.$$

Let Ω_ϵ be the bounded domain defined by $\Omega_\epsilon = \Omega \backslash \overline{B}_\epsilon$. Then the boundary of Ω_ϵ consists of γ and ∂B_ϵ. Let $0 > \mu_1(\epsilon) \geqslant \mu_2(\epsilon) \geqslant \dots$ be the eigenvalues of the Laplacian in Ω_ϵ with the Dirichlet condition on $\gamma \cup \partial B_\epsilon$. And let $0 > \mu_1 \geqslant \mu_2 \geqslant \dots$ be the eigenvalues of the Laplacian in Ω with the Dirichlet condition on γ. We arrange them repeatedly according to their multiplicities.

The main aim of this note is to give an asymptotic expression of $\mu_j(\epsilon)$ when ϵ tends to zero. We have the following two theorems.

THEOREM 1. *Assume that $n = 2$. We fix j. Suppose that μ_j is simple, then the asymptotic relation*

$$\mu_j(\epsilon) - \mu_j = 2\pi(\log\epsilon)^{-1}\varphi_j(w)^2 + 0\big((\log\epsilon)^{-2}\big) \tag{1.1}$$

holds as ϵ tends to zero. Here φ_j denotes the normalized eigenfunction of the Laplacian associated with μ_j, that is, it satisfies

$$\int_\Omega \varphi_j(x)^2\, dx = 1.$$

THEOREM 2. *Assume that $n = 3$. We fix j. Suppose that μ_j is simple, then the asymptotic relation*

$$\mu_j(\epsilon) - \mu_j = -4\pi\epsilon\varphi_j(w)^2 + O(\epsilon^{3/2}) \tag{1.2}$$

holds as ϵ tends to zero. Here φ_j denotes the normalized eigenfunction of the Laplacian associated with μ_j.

The above two theorems are announced in Ozawa [2]. It should be remarked that the remainder terms in (1.1) and (1.2) may not be uniform with respect to j.

In §2, we show that $\lim_{\epsilon\to 0} \mu_j(\epsilon) = \mu_j$ for any j which is a consequence of a result in Rauch-Taylor [4]. In §3, we review the elegant variational formula

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