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ON THE CHOW GROUPS OF CERTAIN RATIONAL SURFACES: A SEQUEL TO A PAPER OF S. BLOCH

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S. Bloch has recently applied the methods of algebraic K-theory to the study of 0-dimensional cycles on rational surfaces, modulo rational equivalence. The best results are obtained for conic bundles over the projective line [1]. In this paper, building upon Bloch's very original ideas and upon some more or less classical facts pertaining to quadratic forms, we shall refine the results of [1], thereby answering some of the questions raised there.

Let k be a perfect field, $\bar{k}$ an algebraic closure of k, and $\mathfrak{g} = \mathrm{Gal}(\bar{k}/k)$. Let X be a *rational*, proper, smooth, geometrically integral variety over k. We denote the function field of X, resp. $\bar{X} = X \times_k \bar{k}$, by $F = k(X)$, resp. $\bar{F} = \bar{k}(X)$. By the very definition of a rational variety, the latter field is purely transcendental over $\bar{k}$. Moreover, for such an X, the $\mathfrak{g}$-module $\mathrm{Pic}\,\bar{X}$ is a free $\mathbf{Z}$-module of finite type: we can regard it as the character group $\hat{S}$ of a k-torus S. Following [1] (as opposed to [2] or [4]) we denote by $A_0(X)$ the group of classes of degree nought 0-dimensional cycles on X with respect to rational equivalence.

In section 1 of this paper, we define a "characteristic" homomorphism

$$\Phi : A_0(X) \rightarrow H^1(k, S)$$

and we show that its image is finite when k is any finitely generated extension of $\mathbf{Q}$. This raises the question: what about the kernel of Φ? Examples with $\dim X \geqslant 3$ suggest one should not expect a general answer, except in the case of *surfaces*.

In this last case, Bloch [1] has produced a K-theoretical interpretation of the kernel and the cokernel of Φ: starting from another definition of Φ, special to dimension 2, he constructs the basic exact sequence:

$$S(k) \rightarrow H^1(\mathfrak{g}, K_2\bar{F}/K_2\bar{k}) \rightarrow A_0(X) \xrightarrow{\Phi} H^1(k, S) \rightarrow H^2(\mathfrak{g}, K_2\bar{F}/K_2\bar{k}). \quad (*)$$

He uses this sequence to show that the image of Φ is finite if k is global, and that the kernel of Φ is finite when X is a conic bundle over $\mathbf{P}^1_k$ and k is local or global. This gives the finiteness of $A_0(X)$ for $X/\mathbf{P}^1_k$ a conic bundle over a local or a global field. He also gets $A_0(X) = 0$ for $X/\mathbf{P}^1_k$ a conic bundle over a C_1-field.

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