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SUBELLIPTIC ESTIMATES AND FAILURE OF SEMI CONTINUITY FOR ORDERS OF CONTACT

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Introduction. The main purpose of this note is to give an example of a pseudoconvex domain for which the maximum order of contact of a complex analytic variety with the boundary of the domain is not an upper semi continuous function. This example has a surprising consequence for subelliptic estimates in the $\bar{\partial}$-Neumann problem. In all examples for which the exact value of epsilon is known, its value turns out to be the reciprocal of Δ; here Δ denotes the maximum order of contact of complex analytic curves with the boundary. See [1,6]. The example presented here shows that this is not generally true; furthermore, a closely related example shows that the best possible epsilon in such an estimate is not stable under deformations of the boundary in any reasonable topology. The motivation for this example comes from the author's papers [2,3,4], and fruitful discussions with Catlin and Kohn.

Preliminaries. Suppose that Ω is a domain in $\mathbf{C}^n$, and that its boundary is a smooth hypersurface. We let r denote a fixed defining function for Ω; it is clear that our notions are independent of the choice of r. We write $M = bd\Omega$, so that $M = r^{-1}(0)$, and now give the definitions of order of contact of complex analytic curves with M at a point p.

Definition 1. If g is a smooth function of a complex variable t, we define $\nu_p(g)$ to be the order of the zero of g at p. We will also use this notation for vector valued functions. Let $\mathcal{C}_p$ denote the set of germs z of holomorphic maps at 0 in $\mathbf{C}$, with values in $\mathbf{C}^n$, such that $z(0) = p$. We write $\mathcal{C}_p^*$ for the elements of $\mathcal{C}_p$ which are not identically constant. We write z^*r for the pullback map $t \to r(z(t))$.

We also need some simple analytic algebra. We let $O = O_p^n$ denote the ring of germs of holomorphic functions at p in $\mathbf{C}^n$. Suppose I is a proper ideal in O. We define an invariant of I as follows.

2.
$$\tau^*(I) = \sup_{z \in \mathcal{C}_p^*} \inf_{g \in I} (\nu(z^*g)/\nu(z)).$$

Here the orders are evaluated at 0. Note that $\tau^*(I)$ is finite precisely when the variety of the ideal consists of p alone. We now give our definition of order of contact. See [2,3,4] for the evolution of this definition.

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