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TRACE FORMULA FOR HILL'S OPERATORS

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1. Introduction. Let $q(x)$ be a real valued C^∞-function of period 1, and let $\lambda_0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \cdots \uparrow\infty$ be the (periodic) spectrum of Hill's operator $Q = -d^2/dx^2 + q(x)$ acting on the class of twice differentiable functions of period 1. As is known, the Θ-series associated with the spectrum

$$\Theta(t) = \sum_{i=0}^{\infty} \exp(-\lambda_i t)$$

has very complicated singularity at $t = 0$, which can usually be described by the asymptotic behavior of $\Theta(t)$ as $t\downarrow 0$. For instance, eliminating the exponentially small terms, one can get the asymptotic expansion into half integral powers of t:

$$\Theta(t) \sim (4\pi t)^{-1/2}\left(1 + \sum_{i=0}^{\infty} A_i t^i\right), \tag{1}$$

in which the coefficients $A_i = A_i(q)$ are non-linear functionals of the potential q of the form

$$A_i(q) = \int_0^1 a_i(q(x), q'(x), \ldots)\, dx.$$

Here a_i is a universal polynomial of $q, q', q'', \ldots$ (see [10] for the proof and basic properties of A_i). In order to take account of exponentially small terms, we first recall the celebrated *Jacobi's inversion formula*

$$\sum_{n=-\infty}^{\infty} \exp(-4\pi^2 n^2 t) = (4\pi t)^{-1/2} \sum_{n=-\infty}^{\infty} \exp(-n^2/4t), \tag{2}$$

in which the left hand side is just the Θ-series corresponding to the case $q(x) \equiv 0$. Since a general Hill's operator can be considered as a perturbation of the one dimensional Laplacian $-d^2/dx^2$, one may expect a "*perturbed*" inversion formula for non-trivial q of the form

$$\Theta(t) = (4\pi t)^{-1/2} \sum_{n=-\infty}^{\infty} \exp(-n^2/4t) F_n(q; t) \tag{3}$$

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